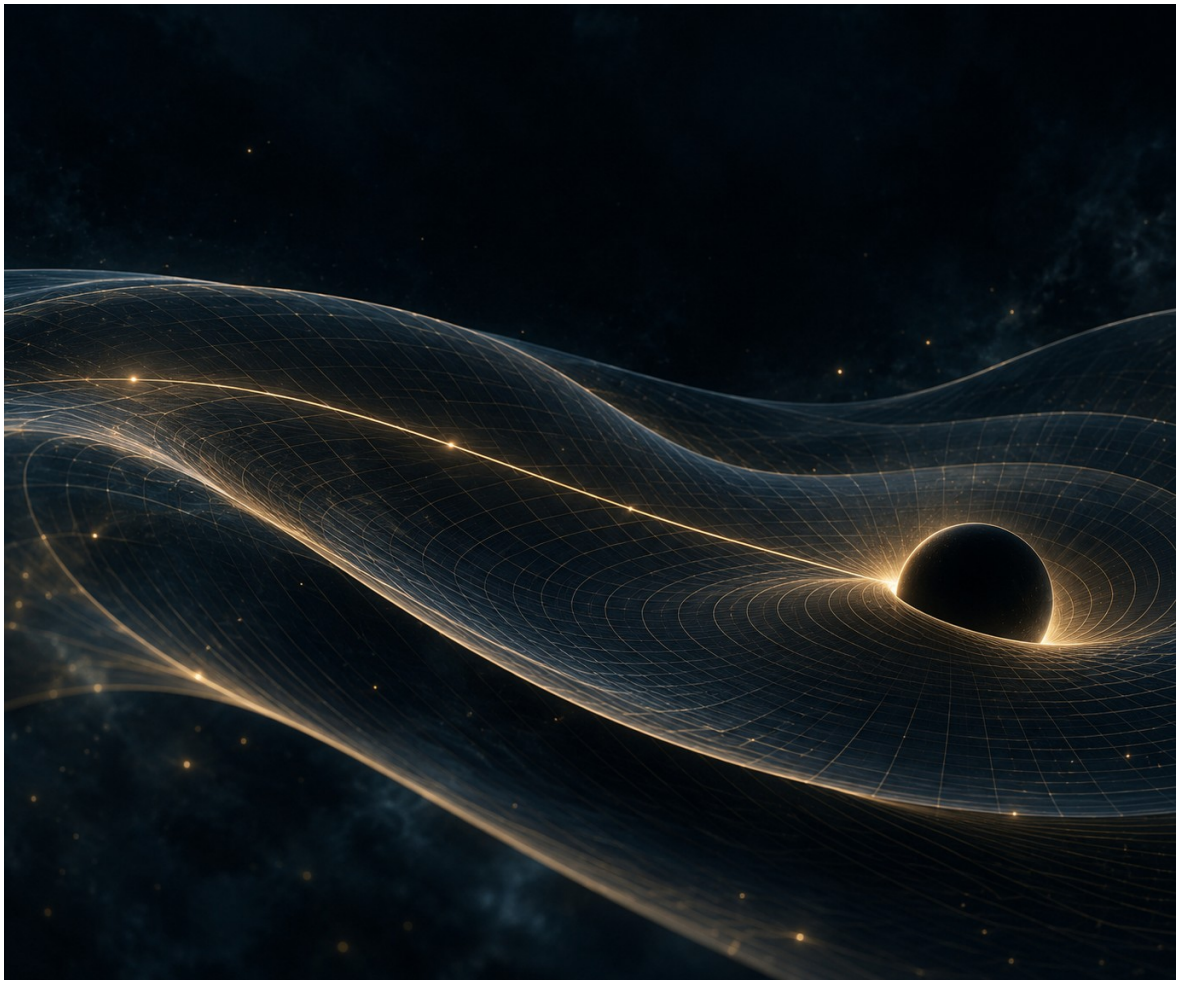
Identifiable Bulk Geometry

Dynamical Completion and Observational Limits in Braneworld Cosmology



Mohammed F. A. Alghenaim

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Publication and Editorial Note

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First book edition, 2026.

This volume develops the supplied research manuscripts into a connected English monograph. Equations use the conventions declared in the text. Numerical examples, synthetic studies and observational comparisons retain their stated evidentiary status. The source register and References distinguish the unpublished manuscript sequence from published primary literature.

Editorial preparation, mathematical cross-checks, computational figures and conceptual cover artwork were assisted by AI tools. The computational companion contains the calculations made for this edition. Scientific responsibility and any subsequent publication decisions remain with the author.

The book includes a linked contents, numbered equations, figure and table lists, a notation guide, five appendices, a glossary, references and a subject index. The companion README records the software required to regenerate its eight scientific figures.

Cover image: conceptual illustration of a brane and surrounding bulk geometry. It is not a numerical spacetime diagram.

Abstract

Cosmological reconstruction is an inverse problem: measured histories constrain a gravitational model, but need not identify each parameter. This book studies a conserved barotropic component on a homogeneous brane in a five-dimensional Einstein bulk. An exact sound history defines a coordinate in which the expansion combines four known functions. Their independence identifies the expansion coefficients. A Riccati equation classifies the exceptional sound histories for which complete functional data retain a degeneracy.

When the effective vacuum amplitude is uncertain, the coefficients determine a continuous family of density normalizations and split brane tensions. Bulk curvature and the geometric dark-radiation coefficient can remain invariant. Absolute matter information restricts this family, but a single density anchor can retain two solutions. The exact inverse and an explicit example distinguish a geometric mass coefficient from a mass defined using a gravitational normalization.

Finite observations require shared density, decline-rate and calibration variables to enforce one history across overlapping intervals. Sound windows constrain integrals and admit frequency aliases; stability requires a declared band and calibration. Statistical validation must account for selection, covariance estimation and the distinction between one observation vector and independent repetitions. Simultaneous outer confidence sets propagate sound, expansion and regularity uncertainty while retaining every admissible normalization.

The action analysis distinguishes vacuum relabeling from classical action scaling, constructs local kinetic realizations, and examines the vacuum defect of a cubic kinetic law and a canonical repair. A scale audit uses the total vacuum stress in the quadratic density correction and separates benchmarks from observational constraints. The account establishes conditional mathematical identification and a dynamical framework. Measuring the required dark-fluid sound, matching local gravity and constructing a complete perturbation likelihood remain explicit physical tasks.

Keywords: braneworld cosmology; inverse problems; identifiability; vacuum normalization; barotropic sound; kinetic scalar fields; statistical calibration; bulk geometry.

Author’s Preface

My aim is to make a demanding question precise: which features of a gravitational model can be learned from cosmological histories? An equation may fit an expansion curve while leaving several physical ingredients interchangeable. Understanding that freedom is part of understanding the theory.

I have organized the account around physical meaning before algebraic technique. A geometrical quantity needs an operational interpretation. A reconstruction needs assumptions, an explicit inverse and a test of uniqueness. A matter history must satisfy conservation throughout one common evolution. Finally, an action must explain what its parameters mean and how a measurement could fix their normalization. Each step deserves its own argument.

The book brings a sequence of research manuscripts into one exposition. Earlier studies investigate finite geometry, matter calibration, sound information and statistical validation. The final study adds covariant normalization, kinetic dynamics and observational scale limits. Their synthesis matters because a weakness at one level can change another result’s interpretation. Exact reconstruction can coexist with normalization freedom; local scalar stability can coexist with an unresolved vacuum limit; a precise fit can coexist with an experiment whose statistical calibration remains incomplete.

I want readers to inspect the results and their limits. Definitions and units are collected for reference. Formulas are numbered, worked examples state their inputs, and the computational record distinguishes calculations made for this edition from results reported in its sources. A claim can thus be traced to the hypothesis or calculation on which it depends.

The intended outcome is a better connection between reconstruction and measurable physics. The book identifies quantities that a future observation model must supply, and freedoms that dynamics or independent calibration might remove. I offer this volume as a foundation for examining that research program, extending it and testing it.

Mohammed F. A. Alghenaim

Introduction to the Book

This volume asks how much bulk geometry a cosmological history determines when matter calibration and vacuum normalization are uncertain. Readers should know calculus, linear algebra and the basic Friedmann description of cosmology. Tensor notation appears in the dynamical chapters, while ordinary differential equations and linear algebra carry the principal inverse.

Chapters 1–3 distinguish observables from parameter labels, state the braneworld conventions and explain finite-epoch reconstruction. Rank supplies a uniqueness condition; projection and short-interval expansions explain why information can remain weak. Chapter 4 requires one common matter history, so finite density values, calibration and sound windows refer to the same conserved fluid.

Chapters 5–6 develop the structural inverse. Sound defines transport functions, making the expansion quadratic in one coordinate plus a dark-radiation term. An exceptional Riccati family marks the failure of complete-history identification. Free vacuum produces a family of physical descriptions. A worked example shows which bulk quantities survive and why one density anchor can be insufficient.

Chapters 7–9 address imperfect information. Sound windows have transfer functions and aliases. Frequency selection changes the statistical problem. Independent training and validation, bounded leakage and covariance calibration give guarantees for specified experiments. Simultaneous outer confidence sets then propagate uncertainty into the physical inverse.

Chapters 10–14 connect the inverse to an action. They separate vacuum bookkeeping from a classical action scale, audit kinetic scalar examples, calculate physical scales and identify the missing links to local gravity and perturbations. Chapter 14 assembles the established results and the measurements needed for progress.

The appendices provide units, reconstruction procedures, a separate scalar–tensor comparison, the computational record and a source guide. The glossary and subject index support later reference. Throughout the book, retain the hypotheses beside the conclusion: an exact history, finite observations, a noisy window and a normalized action are distinct inputs. An inference derives its strength from the information actually available.

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Notation and Conventions

Natural units $c=\hbar=1$ are used unless a numerical conversion states otherwise. The metric signature is $(-,+,+,+)$ on the brane. The reference epoch is $s=0$, with $s=\log(a/a_*)$. A prime on a background function denotes d/ds; a dot denotes proper-time differentiation. Chapter-local indices on statistical matrices do not change these conventions.

Geometry and background reconstruction

Symbol	Meaning and convention
a, a_*	Scale factor and its reference value; the convention used to give them length is stated with the bulk metric
H	Hubble expansion rate
k, ℓ, μ_B	Spatial-curvature index, AdS radius and geometric bulk mass coefficient
M_5, M_4	Five-dimensional Planck scale and reduced four-dimensional background scale; local gravitational matching is specified separately
$\lambda=\lambda_{bare}, U_0$	Split brane constant and constant matter vacuum contribution
$\tau=\tau_{vac}=\lambda+U_0$	Total constant vacuum stress entering the vacuum-subtracted background
$g_{ref}=g_{split}$	Reference split coefficient $\lambda_{bare}/(6M_5^6)$
g_{bg}	Vacuum-subtracted background coefficient $\tau_{vac}/(6M_5^6)$
κ_0, S	Reference normalization $g_{ref}/3$ and a fixed positive curvature scale
h, ν, p	Normalized expansion H^2/S, density $\kappa_0\rho/S$ and pressure $\kappa_0 P/S$ in the inverse chapters
b, u, d	Normalized effective vacuum, spatial curvature and dark-radiation coefficients
η	Inverse split tension $S/(2\kappa_0\lambda_{bare})$
χ	Barotropic derivative $dP/d\rho$; a propagation speed squared only under the declared single-fluid assumptions
q, Q, J	Density decline rate $-\nu'$, sound transport function and its integrated coordinate

Symbol	Meaning and convention
V, C	Density at the reference epoch and the positive transport normalization
α_j, β_j	Fixed-vacuum and free-vacuum expansion coefficients
Σ, L	Free-vacuum discriminant and invariant $L = \Sigma / \left(4\beta_2\right) = 1 / \left(S\ell^2\right)$

Fields, observations and statistical sets

Symbol	Meaning and convention
X	Scalar kinetic invariant $-h^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi/2$, using the induced brane metric
$F\left(X\right)$ or $P\left(X\right)$	A kinetic pressure Lagrangian, with its normalization specified locally
c_ϕ^2	Scalar characteristic sound speed on the stated background
ρ_e, p_e	Excitation density and pressure after removal of the constant vacuum
ϵ_{brane}	Direct quadratic-to-linear ratio $\rho_e / \left(2\tau_{vac}\right)$
Δ, ω, Ω	Window width and frequency variables in logarithmic scale factor
$sinc\, z$	$\sin z / z$, extended continuously to one at zero
A	A design matrix or amplitude, explicitly defined in its chapter
U, P_U	Selected orthonormal subspace basis and its orthogonal projector
C, Σ	Observation covariance matrices when used in statistical chapters; distinguish the local matrix Σ from the scalar discriminant in Chapter 6
α	A significance or error-allocation level in statistical statements; elsewhere a positive action-scaling parameter
n, r, N	Number of independent validation vectors, selected rank and number of expansion epochs, respectively
B_β	An outer coefficient confidence region
I	A physically admissible inverse set, with its conditioning assumptions stated

Statistical covariance matrices are distinct from the scalar discriminant of Chapter 6. Appendix A expands the units crosswalk; Appendix C uses a separate four-dimensional scalar–tensor action.

1 Geometry as an Observable

1.1 The question behind reconstruction

An expanding universe presents an inverse problem. A proposed action and a set of initial conditions predict a geometry. An observed geometry must be interpreted in the opposite direction, through the set of actions, matter histories, calibrations, and initial conditions capable of producing it. The inverse direction can retain freedom even when the forward equations are completely deterministic. This distinction is the organizing question of this book. [1; 2; 3; 4]

Consider two models that give exactly the same expansion rate throughout an interval. If their unobserved matter densities differ, an expansion measurement alone has no means of choosing between them. Increasing its numerical precision does not alter that fact. An additional observation becomes useful when its predicted value changes along the family of equivalent backgrounds. Determining which observations have this property requires an explicit model of both the dynamics and the measurement.

The word *geometry* also requires care. The curvature of a spatial slice, the curvature of an optical metric, the bending of a surface in an ambient space, and the curvature of spacetime are distinct quantities. A visual resemblance between two surfaces does not identify their field equations. We begin with this distinction because every later reconstruction depends on knowing what the geometric input actually means. [1]

1.2 What a geodesic triangle measures

In units with the speed of light equal to one, a homogeneous and isotropic spacetime has the local form

$$d s^2 = - d t^2 + a^2 (t) \left[\frac{d r^2}{1 - k r^2} + r^2 d \Omega^2 \right). \tag{1.1}$$

Here the scale factor has dimensions of length and the comoving coordinates are dimensionless. Once their normalization is fixed, the constant k specifies the intrinsic spatial curvature. On a constant-time slice, every sectional curvature is

$$K_{spatial} = \frac{k}{a^2}. \tag{1.2}$$

A geodesic is defined by the intrinsic metric: its tangent is parallel transported along itself. It need not look straight in an external drawing. For a simple triangle whose sides are geodesics in a smooth, totally geodesic two-dimensional section of the slice, Gauss–Bonnet relates its interior angles to the curvature enclosed by the boundary. If the angles are α, β, γ and the intrinsic area is A, the constant-curvature result is

$$\Delta_{\triangle} = \alpha + \beta + \gamma - \pi = \int_{A} K \, d A = K \, A . \tag{1.3}$$

The assumptions are part of the measurement. A boundary with nongeodesic sides adds geodesic-curvature terms. A region containing a conical singularity needs its singular contribution. A triangle drawn on a projected image does not automatically have the angles, sides, or area required by this formula. [1, Secs. 2–4]

The small-triangle limit makes the sign of curvature tangible. Let u and v be the orthogonal geodesic legs of a small right triangle and w its geodesic hypotenuse. On a sphere of constant positive curvature the exact cosine law is $\cos\left(\sqrt{K}\, w\right) = \cos\left(\sqrt{K}\, u\right) \cos\left(\sqrt{K}\, v\right)$. Expanding both sides and solving to fourth order in the lengths gives

$$w^2 = u^2 + v^2 - \frac{K}{3} u^2 v^2 + O\left(K^2 L^6\right). \tag{1.4}$$

Here L bounds the side lengths and the expansion requires $\left|K\right) L^2 \ll 1$. The hyperbolic cosine law gives the same expression for negative K. Positive curvature decreases the squared hypotenuse relative to its Euclidean value at this order; negative curvature increases it. This is a consequence of the intrinsic metric, independent of how a drawing is arranged on a page. [1, Eq. 7]

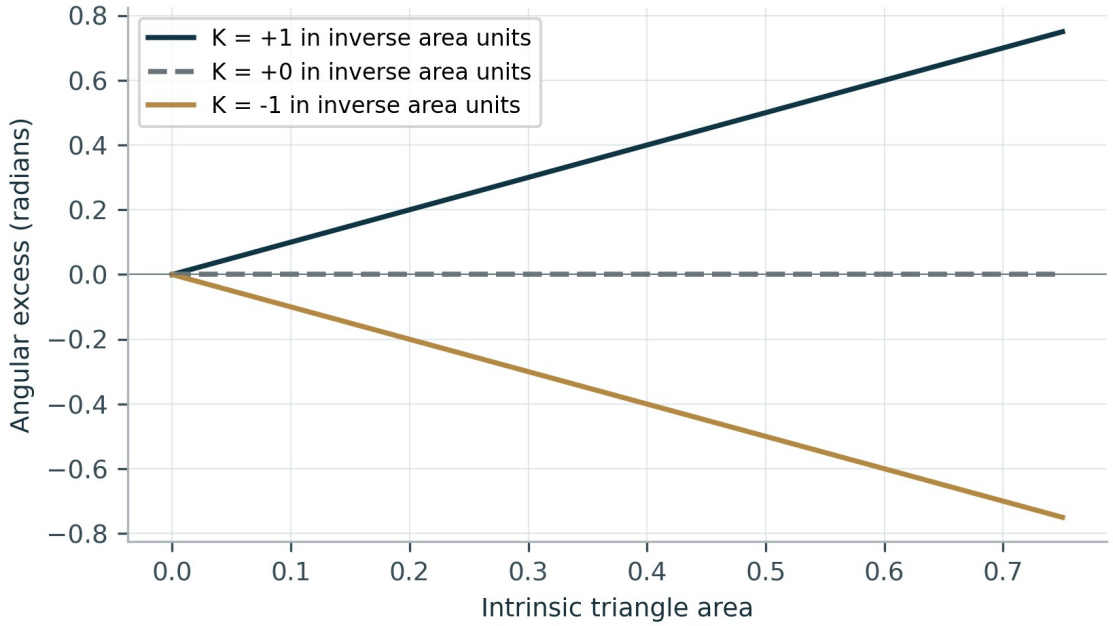

Figure 1.1. Angular excess for three prescribed constant curvatures. The horizontal coordinate is intrinsic triangle area. These exact geometrical relations are illustrations, not astronomical measurements.

1.3 Intrinsic and extrinsic curvature

A cylinder supplies a useful operational example. Its surface bends in ordinary three-dimensional space, yet it can be unrolled onto a plane without stretching. In local coordinates given by distance around the circumference and distance along the axis, its metric is Euclidean. Small geodesic triangles therefore have zero angular excess. The cylinder has extrinsic curvature while its intrinsic Gaussian curvature vanishes. A sphere, by contrast, cannot be flattened without changing distances. [1]

The reverse inference also fails. A positively curved spatial sphere can be embedded as a chosen surface in a flat ambient spacetime; curvature of that surface does not establish curvature of the ambient spacetime. Likewise, an intrinsically flat brane can be embedded in an anti-de Sitter bulk. The brane's extrinsic curvature and the bulk curvature may be nonzero even though its induced background is flat. These examples show why an intrinsic triangle measurement must enter a specified dynamical model before it can constrain an extra dimension. [1, Secs. 4–6]

There is no contradiction with the usefulness of embedding pictures. Such pictures help explain which manifold carries the observers and which carries the

gravitational field. Their role is conceptual. The metric, junction conditions, matter stress, and boundary conditions carry the quantitative content.

1.4 Spatial geometry and optical geometry

For a static metric written as

$$d s^2 = - N^2 (x) d t^2 + h_{i j} (x) d x^i d x^j , \tag{1.5}$$

a clock at rest measures $d \tau = N d t$. The spatial projection of a null path instead obeys a travel-time problem with optical metric

$$d t^2 = \frac{h_{i j}}{N^2} d x^i d x^j . \tag{1.6}$$

The optical metric includes the lapse. Its geodesics are therefore generally different from geodesics of $h_{i j}$. A photographed light-ray triangle and a simultaneous spatial triangle answer different questions. In the Schwarzschild example discussed in the first reference series, the equatorial optical curvature is nonzero and its weak-field behavior is consistent with the usual gravitational deflection. That calculation does not turn every sky triangle into a direct spatial-curvature estimator. [1, Sec. 5]

Even a position-dependent clock rate is insufficient by itself to prove spacetime curvature. Rindler coordinates on a flat spacetime provide a lapse that varies with position for an accelerated congruence. Their Christoffel symbols and relative clock rates can be nontrivial while the Riemann tensor remains zero. The appropriate distinction is between coordinate descriptions, the motion of the observers, and invariant curvature. [1]

These distinctions become especially important in the clock comparisons of Chapter 13 and Appendix C. Clock ratios, orbital accelerations, and lensing can complement one another, but their relation must be derived from the same metric and matter coupling. An approximate potential extracted by one observable cannot silently replace the exact lapse used by another.

1.5 The observational interface

Astronomical observations arrive along our past light cone. A simultaneous spatial slice is a model-dependent reconstruction of those observations. Redshifts, luminosity or angular distances, time delays, and acoustic observables have their own response functions and covariance. A practical analysis must state how those measured quantities constrain H, K, or the residual entering the braneworld equation. [1; 2]

The covariance matters as much as the mean relation. If spatial curvature is inferred from a distance–expansion relation and the same expansion data are then used again in the residual, the resulting quantities are correlated. Treating them as independent adds information that the experiment did not supply. A joint likelihood, or a joint uncertainty region with the appropriate coverage, keeps the dependence explicit. [2, Secs. 2 and 8]

The same discipline applies to sound. The quantity $\chi = d\,p / d\,\rho$ used later belongs to the particular single barotropic fluid entering the adopted background model. A measured acoustic feature in another component is not automatically a measurement of this function. Chapter 7 first studies an explicitly declared window operator; Chapter 13 returns to what an astronomical instrument would have to measure.

1.6 Three levels of inference

We shall distinguish three questions throughout the book. *Compatibility* asks whether at least one allowed model reproduces the data within their uncertainty. *Identification* asks whether the desired parameter or combination has the same value in every compatible model. *Stable recovery* asks whether small measurement changes lead to controlled changes in the recovered quantity. [5; 6]

These are different mathematical properties. A fit with zero residual proves compatibility. Full column rank can establish identification of regression coefficients. A singular value separated from zero supplies a quantitative route toward stability. Physical interpretation adds another step: the fitted coefficients must map into an admissible action and into the physical parameter of interest. Freeing an effective vacuum term will later leave a family of tensions even when the regression coefficients are uniquely determined.

The succeeding chapters follow this sequence. Geometry supplies a well-defined input. A restricted gravitational model supplies a forward relation. Matter and calibration assumptions define the inverse family. Sound, independent normalization, and uncertainty control then determine which conclusions survive. The final part asks whether the reconstructed histories can arise from a predictive and locally consistent dynamics of sufficient magnitude to test.

2 The Braneworld Background and Its Conventions

2.1 The model being reconstructed

The gravitational setting is a thin brane separating two reflection-related regions of a five-dimensional vacuum spacetime. The bulk obeys Einstein gravity with a negative cosmological constant. The brane carries a positive constant tension and a homogeneous, isotropic matter stress. There is no exchange of energy with the bulk in the reconstruction problem. Induced gravity, a Gauss–Bonnet correction, a second brane, and a bulk scalar are absent from this model. The foundational projected equations and cosmological junction dynamics are established results. [7; 8; 9]

This chapter introduces the convention used by the predecessor papers. The symbol λ denotes the split brane tension, also written λ_{bare} when its separation from matter vacuum energy is important. The density ρ includes the matter assigned to the brane sector under that convention. It may therefore include a constant matter contribution that Chapter 10 will separate explicitly. Until that separation is made, a coefficient defined using λ must not automatically be identified with an independently measured local gravitational coupling. [3; 4]

Use $c=\hbar=1$ unless an SI conversion is stated. The four-dimensional signature is $(-,+,+,+)$. The five-dimensional reduced gravitational scale is defined by $8\pi G_5=M_5^{-3}$, and $\Lambda_5=-6/\ell^2$. With an appropriate three-dimensional constant-curvature metric $d\Sigma_k^2$, the Schwarzschild–anti-de Sitter bulk takes the form

$$ds_5^2=-f(r)\,dT^2+\frac{dr^2}{f(r)}+r^2\,d\Sigma_k^2,\quad f(r)=k+\frac{r^2}{\ell^2}-\frac{\mu_B}{r^2}. \tag{2.1}$$

The brane trajectory is $r=a(t)$, where t is its proper cosmological time. The parameters ℓ and μ_B have dimensions of length and length squared, respectively. We take $\mu_B\geq 0$ and retain a fixed normalization of the spatial coordinates. [1; 5]

2.2 From the junction condition to expansion

On the branch with positive total brane density and the declared normal orientation, the spatial junction condition can be written

$$\sqrt{\dot{a}^2+f(a)}=\frac{a(\lambda+\rho)}{6M_5^3}. \tag{2.2}$$

Squaring and dividing by a^2 gives

$$H^2+\frac{k}{a^2}=\frac{(\lambda+\rho)^2}{36M_5^6}-\frac{1}{\ell^2}+\frac{\mu_B}{a^4}. \tag{2.3}$$

The unsquared relation records the branch information lost by squaring. A candidate obtained from the algebra must respect the chosen branch and the physical restrictions. Conservation of the brane matter supplies

$$\dot{\rho}+3H(\rho+p)=0. \tag{2.4}$$

The expansion term proportional to ρ^2 and the bulk term proportional to a^{-4} follow from this established system. Combining the spatial curvature measurement of Chapter 1 with Eq. (2.3) supplies an inference equation; it does not create a new gravitational field equation. [1; 2]

Define the split reference coupling and effective constant by

$$g_{ref}=g_{split}=\frac{\lambda}{6M_5^6},\ \frac{\Lambda_4}{3}=\frac{\lambda^2}{36M_5^6}-\frac{1}{\ell^2}. \tag{2.5}$$

Then

$$H^2+\frac{k}{a^2}=\frac{\Lambda_4}{3}+\frac{g_{ref}\rho}{3}\left(1+\frac{\rho}{2\lambda}\right)+\frac{\mu_B}{a^4}. \tag{2.6}$$

Equivalently, the right side is $\Lambda_4/3+g_{ref}\rho/3+\rho^2/\left(36M_5^6\right)+\mu_B/a^4$. This expanded form makes the dimensions transparent: every term has units of inverse length squared.

The distinction between g_{ref} and a calibrated local $8\pi G_N$ is central to the final manuscript. If a matter constant U_0 is absorbed into the total vacuum stress $\tau=\lambda+U_0$, the vacuum-subtracted background coupling is $g_{bg}=\tau/\left(6M_5^6\right)$. These coefficients coincide in the convention $U_0=0$, but a free vacuum split does not force that convention. Chapters 6 and 10 preserve the predecessor algebra while making the required operational matching explicit. [4, Sec. 4]

2.3 Dimensionless reconstruction variables

Choose fixed reference scales $a_* > 0$ and $S > 0$, with dimensions of length and inverse length squared. They define coordinates and units for inference. Put $\kappa_0 = g_{ref}/3$ and define

$$\begin{aligned} s &= \ln\left(a/a_*\right), h = H^2/S, \\ v &= \kappa_0 \rho / S, \eta = \frac{S}{2\kappa_0 \lambda}, \\ b &= \frac{\Lambda_4}{3S}, u = \frac{k}{a_*^2 S}, \\ d &= \frac{\mu_B}{a_*^4 S}, E(s) = e^{-4s}. \end{aligned} \tag{2.7}$$

Equation (2.6) becomes

$$h(s) = b - u\,e^{-2s} + v(s) + \eta\, v^2(s) + d\,E(s). \tag{2.8}$$

A prime from this point denotes differentiation with respect to s, not proper time. Thus conservation is $v' + 3(v + \varpi) = 0$, where $\varpi = \kappa_0 p / S$. In Chapters 4–9, this dimensionless pressure is denoted by p; physical pressure is distinguished explicitly when units are restored. The dimensionless inverse tension is η. We reserve the symbol τ for total vacuum stress and write η^{-1} explicitly when discussing the inverse of inverse tension. This avoids the different uses of τ in the supplied series and the final action analysis.

At fixed κ_0 and S, the reconstruction parameter maps to

$$\lambda = \frac{S}{2\kappa_0 \eta}, \ell^{-2} = S\left(\frac{1}{4\eta} - b\right). \tag{2.9}$$

Positive finite tension and a negative bulk cosmological constant require $\eta > 0$ and $1/(4\eta) > b$. The boundary $\eta = 0$ describes an infinite-tension limit of the effective reconstruction, not an additional finite-tension brane. Positivity of density and of h must also hold on the interval under consideration. A condition checked only at measured nodes does not prove positivity between them. [10; 11; 5]

Table 2.1. Fixed inputs and the consequences of releasing them. Fixed quantities are assumptions of a calculation, not additional observations. The appropriate row is specified again when a theorem depends on it.

Quantity held fixed	Meaning in the reconstruction	Consequence of releasing it
a_* and S	Coordinate normalization and declared units	Numerical coefficient values and their physical conversion must be transformed consistently
κ_0	The split reference coupling	A new normalization problem enters
b	Effective vacuum contribution under the chosen split	The unique fixed-vacuum tension inverse becomes an equivalence family
u	Spatial-curvature amplitude	An extra nuisance template can absorb a signal direction
d	Bulk term proportional to a^{-4}	Conditional tension sets must be combined through a joint construction or a union

2.4 What it means to vary tension

Holding the split four-dimensional coefficient fixed while varying λ generally changes M_5. Holding an effective constant fixed can also change ℓ. The predecessor comparisons are consequently comparisons within a family of parent theories unless additional microscopic constants have been fixed. They are not automatically different solutions of one action with every coupling held constant. [12; 13; 5]

This is useful rather than merely terminological. An inverse theorem states exactly which ingredients can be inferred within the declared family. A microscopic completion then asks which members of that family a physically normalized action permits. An independent calibration may remove a family direction; naming a coefficient of a scalar kinetic term does not by itself provide that calibration. The distinction will become explicit in the action-scaling orbit of Chapter 10.

2.5 Bulk geometry and the horizon

The bulk coefficient μ_B is a parameter of the metric. Converting it into a physical mass requires the gravitational coupling, the horizon topology, and an appropriate normalization or spatial volume. In a planar bulk one usually discusses a mass density after fixing the planar coordinate normalization. A geometrically identified

μ_B need not therefore identify a unique physical black-hole mass when G_5 varies. [3; 5]

For $k=0$ and $\mu_B>0$, the horizon radius obeys

$$r_h^4=\mu_B\ell^2. \tag{2.10}$$

Writing $L=\ell^{-2}/S>0$, this gives

$$\frac{r_h}{a_*}=\left(\frac{d}{L}\right)^{1/4}. \tag{2.11}$$

The planar restriction belongs beside this formula. For other k, one must solve $f(r_h)=0$ with the corresponding curvature term. For $d=0$ the positive planar Schwarzschild horizon in Eq. (2.10) is absent; the zero limit should not be described as a detected black hole. Exterior-brane restrictions likewise require checking $a(t)>r_h$ on the stated interval if that is the intended branch.

2.6 Background closure and dynamical closure

Equations (2.3) and (2.4), together with a specified homogeneous matter law, can close a background calculation. They do not supply every boundary condition for the coupled perturbations in the bulk and on the brane. The projected Weyl contribution contains information about the bulk gravitational field, and a prediction for structure or the microwave background needs the relevant bulk boundary problem. [14; 4]

The scope of the book is therefore layered. The first parts establish conditional inverse results. The later parts exhibit explicit scalar actions, distinguish normalization freedoms, and test local kinetic conditions and the magnitude of selected effects. A locally positive scalar sound speed is one requirement in this chain. It does not replace a calculation of the full coupled spectrum.

3 Identification from Finitely Many Epochs

3.1 A straight line hidden in the background equation

Suppose first that curvature, the effective vacuum contribution, and the physical density history under the chosen split are supplied. Write $v = m\,e(s)$, where e is a known dimensionless shape and m is its known normalization. With $q_2 = m^2\eta$, subtract the supplied linear and geometric terms from Eq. (2.8). The remaining response is

$$y(s) = q_2 e^2(s) + d\,e^{-4s}. \tag{3.1}$$

Multiplying by e^{4s} gives a line:

$$z(s) = q_2 x(s) + d\,,\; x(s) = e^{4s}e^2(s)\,,\; z(s) = e^{4s}y(s). \tag{3.2}$$

Two epochs with different x determine the slope and intercept. Any additional epoch supplies a compatibility check. Explicitly,

$$q_2 = \frac{z_j - z_i}{x_j - x_i}\,, d = \frac{x_j z_i - x_i z_j}{x_j - x_i}. \tag{3.3}$$

This result requires the denominator to be nonzero. If every sampled x_i is identical, compatible data must have every z_i identical, and only the combination $q_2 x_i + d$ is identified. Determinantal contrasts that vanish automatically at the wrong assumed rank cannot replace this condition. [2, Sec. 3]

The construction is derivative-free at the level of supplied residual values. It says nothing by itself about the measurement procedure used to construct those values. It also exposes a practical limitation: as $x_j - x_i$ becomes small, finite errors can generate large changes in the inferred slope even though the exact two-point formula remains valid.

3.2 Nuisance directions and exact degeneracy

The line is a special case of a linear model with nuisance terms. Write the finite response vector as

$$y = N\,b + q_2 f + \epsilon\,, f = \left(e_i^2\right)_i. \tag{3.4}$$

The columns of N contain the geometric or bulk terms whose amplitudes are free. They are e^{-4s} when only the bulk amplitude is free; $1\,, e^{-4s}$ when the effective

vacuum is also free; and $1, e^{-2s}, e^{-4s}$ when spatial curvature is free as well. A supplied quantity with measurement uncertainty belongs in the joint model or its covariance rather than being treated as exact.

Proposition 3.1. In the interior of the physical domain, the quadratic coefficient is unidentified precisely when its sampled column lies in the span of the nuisance columns.

Proof. Subtract two compatible coefficient representations. If their quadratic coefficients differ, the subtraction expresses f as a nuisance combination. Conversely, any such representation lets a change in q_2 be absorbed by a corresponding change in the nuisance amplitudes. Sufficiently small changes preserve strict physical inequalities at an interior point. Boundary restrictions may need a separate analysis. The same argument works for functions observed throughout an interval. ▫ [2, Proposition 3]

For one conserved fluid with constant $w = p/\rho$, the density behaves as $e = e^{-3(1+w)s}$. Its square has exponent $v = 6(1+w)$. Distinct real exponential functions are linearly independent, and their evaluation matrix has full rank at sufficiently many distinct epochs. One proof divides a supposed linear combination by one exponential, differentiates, and uses Rolle's theorem inductively to bound its number of zeros. The full constant-fluid classification is then immediate. [2, Appendix B]

Table 3.1. Exact nuisance degeneracies for a supplied density normalization. The density normalization is supplied in this table. Exactly enough points to determine all amplitudes leave no residual degree of freedom for a goodness-of-fit test.

Free nuisance templates	Values of w that hide q_2	Generic epochs to fit and to test
e^{-4s}	$-1/3$	2 and 3
$1, e^{-4s}$	$-1, -1/3$	3 and 4
$1, e^{-2s}, e^{-4s}$	$-1, -2/3, -1/3$	4 and 5

A changing equation of state need not restore identification. With a free effective vacuum and bulk amplitude, any positive history satisfying $e^2 = A + B e^{-4s}$ lies in the nuisance span. Conservation gives a generally varying $w(s)$, yet the quadratic term is absorbed exactly into new constant and radiation-like amplitudes. Some such histories fail a barotropic sound condition on part of their domain. Their role at this stage is to demonstrate background degeneracy; Chapter 4 imposes a definite matter class. [2, Sec. 5]

3.3 Correlated errors and projected information

Assume an exact design, a known positive-definite covariance $\Sigma = C\,C^{T}$, and Gaussian response errors. Whiten all vectors by C^{-1}. Let P_N and P_X be orthogonal projections onto the whitened nuisance design and the full design $X = [N, f]$, respectively. Define

$$v_f = \left(I - P_N\right) C^{-1} f,\; I_{q_2} = v_f^{T} v_f. \tag{3.5}$$

When $I_{q_2} > 0$, the unconstrained estimate and its variance are

$$\hat{q}_2 = \frac{v_f^{T} C^{-1} y}{I_{q_2}},\; Var\left(\hat{q}_2\right) = I_{q_2}^{-1}. \tag{3.6}$$

The residual statistic obeys

$$T = \left\| \left(I - P_X\right) C^{-1} y \right)^{2} \sim \chi^{2}_{n-r},\; r = rank\, X, \tag{3.7}$$

under that unconstrained linear Gaussian model. Whitening turns the noise covariance into the identity, and projection leaves $n - r$ independent standard-normal coordinates. This proves the stated distribution. The information in Eq. (3.5) is the part of the target column that survives nuisance removal. [2, Proposition 2]

A deterministic omitted component can bias a coefficient while remaining completely hidden in the fitted column space. If its whitened vector is in that space, the residual test has no noncentrality from that component. Passing the test establishes compatibility with a finite-dimensional mean model; it does not assign a unique gravitational origin to its coefficients.

The distribution in Eq. (3.7) does not automatically survive positivity constraints, uncertain density regressors, a covariance estimated from the same data, or a frequency selected by searching the data. Chapters 8 and 9 address specific replacements under additional assumptions. Their procedures retain different targets and should not be mixed into one unqualified calibration claim.

3.4 Why a short interval can contain little information

Take fixed sampling coordinates t_i and set $s_i = L\,t_i$. Let the target be $e^{-\nu L t_i}$, with k nuisance exponents $s_1, \ldots, s_k$ that are distinct from one another and from ν. For independent absolute response error σ, the leading conditional information is

$$I_{q_2} = \frac{L^{2k}}{\sigma^{2}\left(k!\right)^{2}} \prod_{j=1}^{k} \left(\nu - s_j\right)^{2} \left\| \left(I - P_{k-1}\right) t^{k} \right)^{2} + O\left(L^{2k+1}\right). \tag{3.8}$$

Here P_{k-1} projects onto sampled polynomials through degree $k-1$, and $n \geq k+1$ distinct sampling coordinates are assumed. To see the origin of the result, expand each exponential in powers of L. Interpolation in the exponent variable constructs a nuisance combination matching the target through order $k-1$. The first unmatched coefficient is the product $\prod_j (\nu - s_j)/k!$. Projecting its sampled degree-k polynomial off the lower-degree space and squaring the remaining norm gives Eq. (3.8). [2, Proposition 4 and Appendix C]

One, two, and three nuisance powers therefore suppress the information as L^2, L^4, L^6 under this fixed experiment. An exact exponent collision makes it zero rather than merely smaller. These scalings hold with fixed sample count, sample coordinates, and absolute error; they are not a survey forecast in which exposure, selection, or noise changes with interval length.

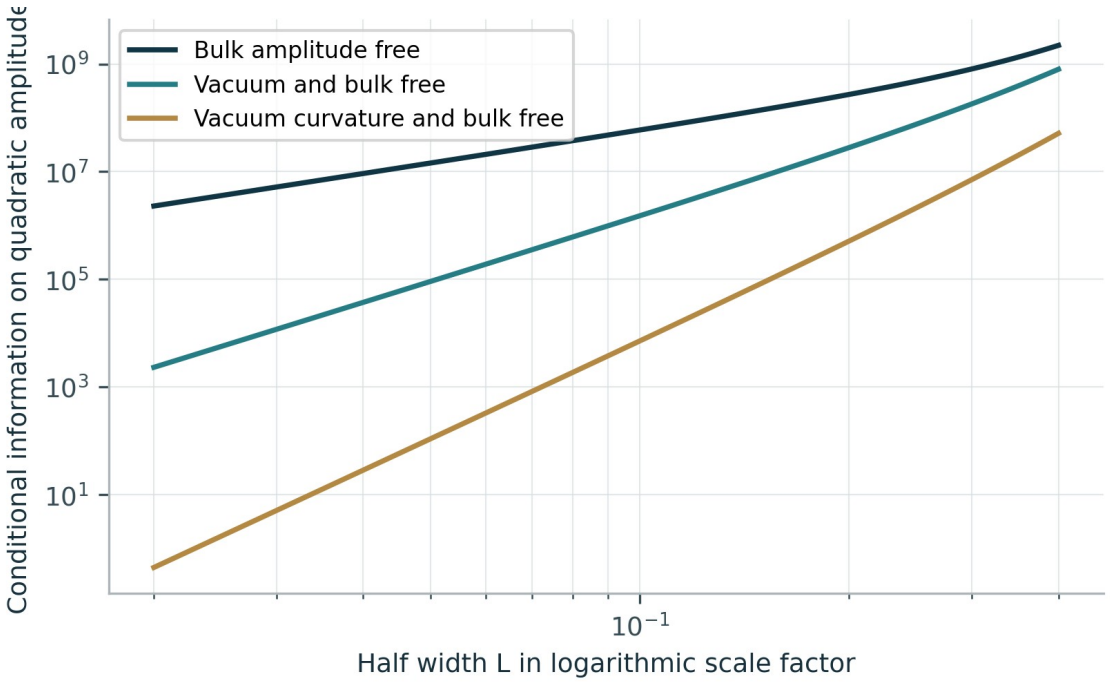

Figure 3.1. Conditional information for a prescribed dust-density template as the logarithmic interval changes. The QR-projected design uses 41 fixed relative epochs and known absolute errors. The curves illustrate information loss from nuisance fitting, not an observational sensitivity forecast.

3.5 Constant density gain and a tension invariant

Now suppose that a positive proxy for the total modeled density has an unknown constant multiplicative gain: $z = g\,v$ with $g > 0$. After supplying or fitting the appropriate nuisance terms, the response takes the form

$$y = A\,z + B\,z^2 + N\,\beta\,,\; A = g^{-1},\; B = \eta\,g^{-2}. \tag{3.9}$$

If the columns $z\,, z^2\,, N$ have full rank, the fitted coefficients determine

$$\eta = \frac{B}{A^2}\,,\; \lambda = \frac{S}{2\kappa_0}\,\frac{A^2}{B}. \tag{3.10}$$

A constant rescaling of the proxy changes A by the inverse scale and B by its inverse square; the ratio is unchanged. This is a conditional algebraic invariant. It is not an assertion that the ratio of noisy estimates is unbiased. It also requires a proxy for the total density entering the model, not merely the density of a visible tracer or one component of a mixture. [12, Proposition 1]

Unknown gain creates an additional radiation collision. For $w = 1/3$, the linear matter term and the free bulk term both vary as e^{-4s}. The coefficient of z^2 can still be separated, but the coefficient of z cannot. A quadratic correction may thus be identified while the tension ratio is not. Fixing the bulk amplitude independently removes that particular coincidence. [12, Sec. 4]

The cancellation is also specific to constant gain. A gain that changes freely with epoch can absorb changes in inverse tension by changing the inferred density history. Chapter 4 considers the intermediate case in which the logarithmic gain has a declared bound on its variation.

3.6 Confidence regions for a nonlinear ratio

The ratio in Eq. (3.10) must be treated globally when its denominator is uncertain. Under the exact-design Gaussian model, a joint confidence ellipse for $(A\,, B)$ has known coverage. Map its part with $A > 0$ and $B > 0$ through A^2/B to obtain a tension-scale confidence set. Whenever the joint ellipse contains the true pair, its image contains the true ratio. The projected set therefore has at least the ellipse's coverage. [12, Sec. 6]

If the ellipse crosses $B = 0$ at a point with positive A, its image has no finite upper tension endpoint. A small local error estimate obtained by differentiating the ratio does not contradict this fact. The supplied reference gives a concrete example: a local standard deviation of $\log\left(A^2/B\right)$ near 0.0508 coexists with an unbounded

upper endpoint of the projected 95 percent set. Both statements describe the same synthetic mean-data design, but only the global set records its denominator uncertainty. [12, Table 3]

Uncertain density measurements introduce another distinction. A joint likelihood can treat the true proxy values as latent quantities, with correlations between geometry and density measurements retained. Its local Schur-complement information is useful, but it is not a substitute for the full nonlinear likelihood or a finite-sample coverage argument. Profiling latent values and integrating them against a population model are different inferential operations. [12, Sec. 7]

3.7 From a fitted coefficient to a physical claim

The results of this chapter identify conditional linear directions and, when the gain assumptions permit it, a ratio. They do not establish a unique higher-dimensional origin. A four-dimensional effective matter description can reproduce the same background terms. For instance, the quadratic contribution of a constant-w fluid has an effective background equation of state $w_Q = 1 + 2\,w$, while the bulk a^{-4} term behaves like radiation in the background equation. Such a construction need not reproduce every perturbation. Its existence is already enough to delimit what background data alone can prove. [2, Proposition 5]

The next step is to insist that all allowed observations and matter conditions belong to one history. This requirement changes the mathematics. Separate sharp bounds can have an intersection even when no common physical realization exists. Chapter 4 formulates the shared-history problem before Chapters 5 and 6 ask which combinations of the bulk geometry remain identifiable.

4 Matter Histories and Common Reconstruction

4.1 What a common history must satisfy

An expansion curve does not determine its matter interpretation until the gravitational coefficients, the density calibration and the allowed equation of state have been specified. A candidate must satisfy these requirements through one history. Checking each requirement with a different interpolating curve can accept data that no physical history realizes. This chapter develops the distinction first for a complete residual curve and then for finite values and sound windows. [10; 13; 11; 15]

Use the normalization of Chapter 2, with fixed S, a_* and $\kappa_0 = g_{ref}/3$. Throughout this chapter the effective vacuum, curvature and dark-radiation coefficients b, u, d are supplied. Define

$$\begin{gathered} s = \log\left(a/a_*\right), h = H^2/S, \\ B(s) = b - u e^{-2s} + d e^{-4s}, \\ R = h - B = \nu + \eta \nu^2, \\ \nu = \kappa_0 \rho / S, \eta = \frac{S}{2\kappa_0 \lambda_{bare}}. \end{gathered} \tag{4.1}$$

The branch has positive density, positive constant tension, one conserved barotropic fluid and no exchange with the vacuum bulk. Finite physical models have $\eta > 0$ and $\eta b < 1/4$. We sometimes include $\eta = 0$ as the general-relativistic, infinite-tension boundary. It is not an additional finite-tension model. The auxiliary condition $B \geq 0$ guarantees positive expansion for the interpolations below; it is sufficient rather than necessary. Releasing a supplied background coefficient requires a joint analysis or a union of conditional sets. Chapter 6 treats the distinct free-vacuum problem. [11; 3]

4.2 Density calibration as a functional constraint

For a proposed inverse tension, the positive density is uniquely

$$f_\eta(R) = \frac{2R}{1 + \sqrt{1 + 4\eta R}}, f_0(R) = R. \tag{4.2}$$

Suppose a proxy measures that same density with log gain γ: $y = \log z = \log \nu + \gamma$. An external calibration specification supplies $\left|\gamma(s) - \gamma(t)\right| \leq \delta\left|s - t\right)$. The bound controls

drift; it leaves the absolute gain free. An arbitrary changing gain can absorb a different density history, whereas a specified drift bound can exclude some histories. [12; 10]

Let $F_\eta(R)=\log f_\eta(R)$. For $X>Y>0$, the contrast $F_\eta(X)-F_\eta(Y)$ decreases strictly as η increases. The key identity is

$$\partial_R \partial_\eta F_\eta(R) = -(1+4\eta R)^{-3/2} < 0. \tag{4.3}$$

Integrating over $[Y, X]$ proves the assertion. The contrast decreases from $\log(X/Y)$ to $1/2\log(X/Y)$. If an interior contrast equals c, writing $r=e^c$ gives its unique positive solution

$$\eta = \frac{r(r-1)(X-rY)}{(r^2Y-X)^2}. \tag{4.4}$$

Endpoint contrasts must be handled by limits. The physical upper bound on η, when present, is imposed separately. This monotonicity explains why the calibration-only identified set is an interval; connectedness is not a general property of every later reconstruction problem. [10]

For rectangular residual and proxy bounds, the possible gains at epoch s_i lie in

$$\begin{aligned} \gamma_i^- &= y_i^- - F_\eta(R_i^+), \\ \gamma_i^+ &= y_i^+ - F_\eta(R_i^-). \end{aligned} \tag{4.5}$$

Proposition 4.1 — Compatible gain values. A gain satisfying all intervals and the drift bound exists exactly when

$$\gamma_i^- - \gamma_j^+ \le \delta\,|s_i - s_j| \text{ for all } i, j. \tag{4.6}$$

Proof. Necessity follows by inserting any permitted gain values. For sufficiency, the envelope

$$\gamma(s) = \max_i \{\gamma_i^- - \delta\,|s - s_i|\} \tag{4.7}$$

is δ-Lipschitz and lies in every required gain interval at its node. The monotone inverse then supplies corresponding residual and proxy values. Each inequality in (4.6) is a monotone contrast constraint, so their intersection with the physical domain is an interval. ▫ [10]

All pairs matter when interval values can be selected. At epochs $0, 1, 2$, take gain intervals $[0,0], [0,2), [2,2)$ and $\delta = 1/2$. Either adjacent pair can be fitted separately. The endpoint gains nevertheless differ by two over a distance of two, exceeding the allowed difference of one. Adjacent inequalities suffice only if they act on one shared vector of unknown gains, or if the exact gains have already been fixed.

4.3 Causal matter on a complete residual curve

For a positive C^2 residual with $R'<0$, conservation fixes pressure after $v=f_\eta(R)$ is chosen. With $p=\kappa_0 P/S$,

$$\begin{gathered} v'=-3(v+p),\, q=-v'>0, \\ p=-v+q/3, \\ \chi_\eta=\frac{p'}{v'}=-1-\frac{R''}{3R'}+\frac{2\eta R'}{3(1+4\eta R)}. \end{gathered} \tag{4.8}$$

Here χ is the barotropic sound speed squared in units with $c=1$. For a mixture with entropy perturbations, the ratio of background derivatives need not equal a propagation speed. The single-fluid assumption is therefore substantive. [13]

The restrictions $0\le\chi_\eta\le 1$ become affine inequalities in the candidate parameter:

$$\begin{gathered} A_3+\eta B_3\ge 0,\, A_6+\eta B_6\le 0, \\ A_j=R''+jR', \\ B_j=4RA_j-2(R')^2. \end{gathered} \tag{4.9}$$

The negative sign of R' reverses the inequalities during multiplication. Intersecting these half-lines over the whole interval gives the exact conditional matter set. Moreover, $\partial_\eta\chi_\eta=2R'/\left[3(1+4\eta R)^2\right]<0$. A dust-generated residual, $R=e^{-3s}+\eta_0 e^{-6s}$, consequently allows the entire range $0\le\eta\le\eta_0$ under this matter restriction. Causal matter narrows an ambiguity without necessarily removing it. Differentiating a noisy interpolation and applying (4.9) does not inherit this complete-curve theorem. [13]

4.4 Finite values and exact reachable drops

Finite observations require no measured derivatives. Treat the node decline rates q_i as latent variables and impose the differential inequalities implied by a sound margin:

$$\begin{gathered} m\le\chi\le 1-m,\, 0\le m<1/2, \\ -Bq\le q'\le -Aq,\, A=3(1+m),\, B=3(2-m). \end{gathered} \tag{4.10}$$

On a cell of length Δ, positive endpoint rates must satisfy $e^{-B\Delta}q_0\le q_1\le e^{-A\Delta}q_0$. Every admissible rate lies between

$$\begin{aligned} q_-(t)&=max\left\{q_0e^{-Bt},\, q_1e^{A(\Delta-t)}\right\}, \\ q_+(t)&=min\left\{q_0e^{-At},\, q_1e^{B(\Delta-t)}\right\}. \end{aligned} \tag{4.11}$$

Theorem 4.2 — Reachable density drop. The prescribed transition exists in the class with absolutely continuous $q>0$ exactly when its drop $D=v_0-v_1$ satisfies

$$L_m=\int_0^{\Delta} q_-\,dt\le D\le\int_0^{\Delta} q_+\,dt=U_m. \tag{4.12}$$

Proof. The differential bounds integrated forward and backward give (4.11), proving necessity. Both envelopes satisfy the same linear inequalities and endpoint values. Their convex combination remains admissible and attains every integral between L_m and U_m. Integrating that rate constructs the required positive decreasing density. ▫ [11; 16]

For $m=0$, direct integration yields

$$\begin{aligned} 6L_0&=q_0-2q_1+e^{6\Delta}q_1^2/q_0,\\ 6U_0&=2q_0-q_1-e^{-6\Delta}q_0^2/q_1. \end{aligned} \tag{4.13}$$

Let $Q=\{(a,b,c):a,b\ge 0,ab\ge c^2\}$. The drop inequalities are equivalently the two rotated-cone conditions

$$\begin{aligned} &\left(q_0,6D-q_0+2q_1,e^{3\Delta}q_1\right)\in Q,\\ &\left(q_1,2q_0-q_1-6D,e^{-3\Delta}q_0\right)\in Q. \end{aligned} \tag{4.14}$$

For positive margins, the analogous representation uses power cones. One shared q_i must serve both cells meeting at a node. This yields a continuous rate and a C^1 density, hence a single pressure law. At fixed η, residual intervals become density intervals through (4.2), while proxy restrictions become the linear inequalities

$$v_j\le\exp\{\delta\,|\,s_i-s_j)-y_i^-+y_j^+\}\,v_i. \tag{4.15}$$

Together these form a convex feasibility problem in the shared v_i,q_i. Convexity at each fixed η does not establish that its projection over η is one interval. A strictly separated first and last density interval excludes the zero-rate branch. Without that separation, strict positive enthalpy must be established explicitly. [11; 16]

The need for a common history has an exact counterexample. At spacing $\Delta=\log 2/3$, choose

$$(v_0,v_1,v_2,v_3)=\left(\frac{21}{8},\frac{13}{8},\frac{9}{8},1\right),(D_0,D_1,D_2)=\left(1,\frac{1}{2},\frac{1}{8}\right). \tag{4.16}$$

For three equally spaced epochs the necessary and sufficient condition is $e^{-6\Delta}\le D_1/D_0\le e^{-3\Delta}$. Both adjacent triples in (4.16) pass. The first saturates the upper bound, forcing $q'=-3q$ throughout its two cells; the second saturates the lower

bound, forcing $q' = -6q$ throughout its two cells. Their overlap cannot satisfy both with $q > 0$. Matching density values has concealed incompatible pressures. [11]

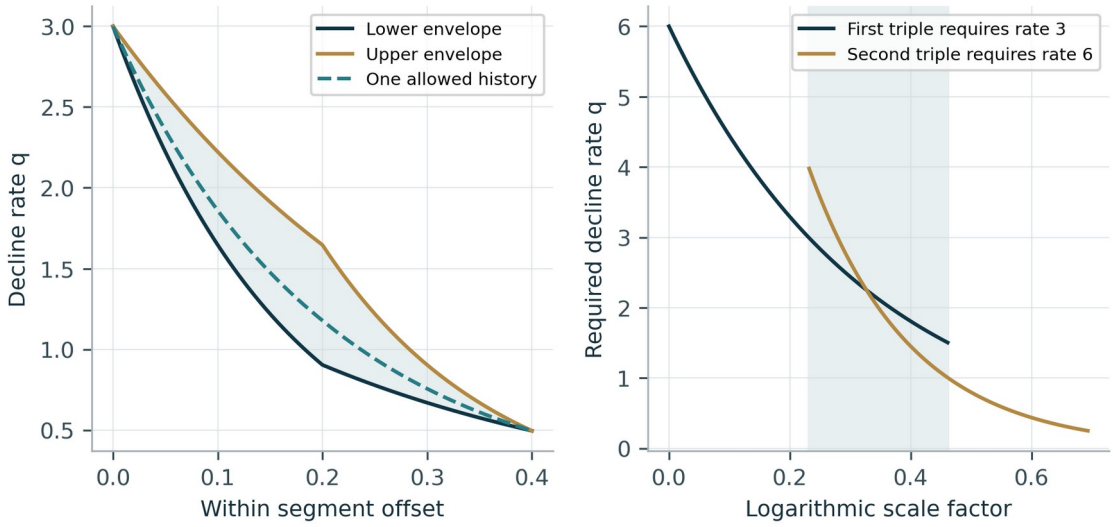

Figure 4.1. Left: exact lower and upper decline-rate envelopes, together with one allowed convex interpolation. Right: the two locally feasible adjacent triples of Equation 4.16 require incompatible rates on their shaded overlap. All curves are constructed examples.

The same example separates marginal from joint calibration tests. Give each density a box of $\pm 20\%$, let the exact proxies equal (4.16), and demand constant gain. Calibration alone fits the central values. Matter alone fits $v(s) = 7/8 + (7/4)e^{-3s}$. Joint feasibility would force all densities to be one common multiple of (4.16), preserving its contradiction. The intersection of two separately projected parameter sets can therefore be strictly larger than the common-history set. [11]

4.5 Smooth completion and the information in anchors

A weak interpolant may have jumps in χ. If a feasible history has a positive margin m, it admits a smooth completion through the same finite density values with any smaller positive margin $m' < m$. Positive convolution preserves the linear rate inequalities; small smooth corrections supported inside the cells restore the exact density integrals. The relaxed margin supplies the slack needed by those corrections. When prescribed sound windows are also retained, node corrections first restore their endpoint rates. This theorem preserves finite constraints, not the original full curve or necessarily the extremal margin. [16; 15]

Every such strictly positive sound history has a local purely kinetic scalar representation. In dimensionless pressure units, set

$$X(s)=X_{*}\exp\left[-6\int_{s_{*}}^{s}\chi(t)\,dt\right],$$
$$P(X(s))=p(s),\ P_{X}=\frac{q}{6X}>0, \tag{4.17}$$
$$2X\,P_{X}-P=\nu,\ P_{X}+2X\,P_{XX}=\frac{P_{X}}{\chi}>0.$$

Strict monotonicity of X makes the definition single valued. The physical pressure Lagrangian is $(S/\kappa_{0})P$; X_{*} sets a field normalization. This construction proves local existence on the sampled kinetic range. It supplies neither a preferred action nor its extension to $X=0$, absolute matter calibration, ultraviolet control or stability of the coupled brane–bulk system. Chapter 10 examines those distinctions at action level. [16; 4]

Smoothness alone also leaves surprising freedom. Any finite collection of interior point values of χ in $(0,1)$, at distinct epochs, can be imposed while preserving finite density nodes, provided some positive margin remains available and no derivative bound is specified. Narrow smooth patches set the sound values; their integrated effects can be corrected away from the anchors. Shrinking a patch makes its derivative larger, so this argument does not survive an externally fixed regularity budget. [17]

For $|\chi(s)-\chi(t)|\le L|s-t|$ and compatible anchors $\chi(r_{j})=a_{j}$, the extremal profiles are

$$\chi_{-}(s)=\max\{m,\max_{j}(a_{j}-L|s-r_{j}|)\},$$
$$\chi_{+}(s)=\min\{1-m,\min_{j}(a_{j}+L|s-r_{j}|)\}. \tag{4.18}$$

Writing $Q_{\chi}(s)=\exp\left[-3(s-s_{0})-3\int_{s_{0}}^{s}\chi\right]$, three density nodes are compatible exactly when their drop ratio lies between the ratios of $\int_{s_{1}}^{s_{2}}Q_{\chi}$ to $\int_{s_{0}}^{s_{1}}Q_{\chi}$ for χ_{+} and χ_{-}, respectively. If one sound profile is larger, the ratio of its Q to the other decreases with s, reducing this later-to-earlier integral ratio. Convex interpolation of the envelopes covers the intermediate ratios continuously. This result applies to three nodes; additional nodes must still share one profile. [17]

4.6 Windows constrain integrals of the same history

A uniform sound mean over $[a,b)$ has an exact endpoint interpretation:

$$\overline{\chi}_{[a,b)} = -1 - \frac{\log q(b) - \log q(a)}{3(b-a)}. \tag{4.19}$$

Thus a mean interval $[l,u)$ gives linear rate-ratio bounds $e^{-3(1+u)(b-a)} q(a) \le q(b) \le e^{-3(1+l)(b-a)} q(a)$. Add every window endpoint to the common node set and retain the same density, rate and gain variables. Two adjacent equal-width means of 0.2 and 0.4 force the union mean to be 0.3; a reported union mean of 0.5 cannot be repaired by another matter interpolation. [15]

Exact adjacent means fix rate ratios, leaving one common rate amplitude. On a fixed compact physical search interval, exact ordered residuals and the supplied background produce sharp interval bounds on η. Consider nested meshes on a fixed epoch interval whose largest spacing tends to zero. If every mesh uses exact adjacent sound means and exact samples from one positive strictly decreasing continuous residual curve generated by an admissible history, the compatible intervals shrink to that history's inverse tension. Fixed nonzero sound uncertainty can preserve a neighborhood of indistinguishable values on every mesh. Refining sampling and reducing uncertainty are different operations. [15]

Table 4.1. Information inputs and the scope of common-history results.

Input	Exact conclusion in its stated class	Essential qualification
Complete residual and matter inequalities	Conditional interval of inverse tensions	Requires control between all epochs
Finite values with shared rates and gains	Exact common-history feasibility at fixed η	Its parameter projection need not be an interval
Smoothness and isolated sound anchors	Local smooth completions remain plentiful	No fixed derivative budget
Finite sound windows	Shared endpoint constraints on q	The mean is of χ in $\log a$
Simultaneous measurement region	Coverage transfers through feasible histories	A deterministic box alone has no probability level

These sets describe identification under the stated inputs. Confidence coverage additionally requires a simultaneous measurement event; tension in physical units requires matched density and gravitational conventions. The following chapters retain these requirements while parametrizing the sound history.

5 Sound Coordinates and Exceptional Histories

5.1 Conservation supplies a useful coordinate

Sound information changes the reconstruction problem because it constrains how a conserved fluid moves through its possible density states. The relevant quantity is the barotropic derivative $\chi = d\,p/d\,v$. It is not the ratio p/v, and it is not the sound horizon inferred from baryon acoustic oscillations. Throughout this chapter there is one conserved barotropic component, positive density and enthalpy, constant positive brane tension, and no energy exchange with the vacuum bulk. The reference scales S, a_* and coefficient $\kappa_0 = g_{ref}/3$ remain fixed. The effective vacuum amplitude b is initially supplied; the next chapter releases it. [13; 18; 19]

Let $I = [s_-, s_+)$ be a compact interval containing zero, with $s = \log(a/a_*)$. Conservation and the definition $q = -v' > 0$ give

$$v' = -3(v+p),\ p = -v + \frac{q}{3},\ q' = -3(1+\chi)q. \tag{5.1}$$

For continuous $0 < \chi < 1$, introduce

$$\begin{gathered} Q(s) = \exp\left[-3s - 3\int_0^s \chi(t)\,dt\right], \\ J(s) = -\int_0^s Q(t)\,dt, \\ q(s) = C\,Q(s),\ v(s) = V + C\,J(s),\ C > 0. \end{gathered} \tag{5.2}$$

Here $Q(0) = 1$, $J(0) = 0$, and $V = v(0)$. Since $J' = -Q < 0$, the sound history provides a one-to-one coordinate on the interval. It determines the shape of the transport, while V and C remain free. In particular, specifying $d\,p/d\,v$ leaves an additive pressure constant undetermined.

Write $E = e^{-4s}$ and remove the supplied vacuum and curvature terms from the dimensionless expansion:

$$\begin{gathered} F = h - b + u\,e^{-2s} = v + \eta\,v^2 + d\,E, \\ \eta = \frac{S}{2\kappa_0 \lambda_{bare}} > 0, \\ F = \alpha_0 + \alpha_1 J + \alpha_2 J^2 + d\,E, \\ \alpha_0 = V + \eta V^2,\ \alpha_1 = C(1 + 2\eta V),\ \alpha_2 = \eta C^2. \end{gathered} \tag{5.3}$$

This is the four-column inverse: the nonlinear matter history enters through the known coordinate J, leaving a linear coefficient problem. Its fixed-vacuum physical inverse is

$$\begin{aligned} \Delta_\alpha &= \alpha_1^2 - 4\,\alpha_0\alpha_2 = C^2, \\ C &= \sqrt{\Delta_\alpha}, \eta = \frac{\alpha_2}{\Delta_\alpha}, \\ V &= \frac{2C\,\alpha_0}{\alpha_1 + C}, \lambda_{bare} = \frac{S\,\Delta_\alpha}{2\kappa_0\alpha_2}. \end{aligned} \tag{5.4}$$

The positive branch requires $\alpha_0, \alpha_1, \alpha_2, \Delta_\alpha > 0$, $d \geq 0$, positive density and expansion throughout I, and $1/(4\eta) - b > 0$. Algebraic uniqueness is therefore separate from physical existence. These conditions also identify what is held fixed when the word *tension* is used in this reconstruction. The physical vacuum convention is revisited in Chapters 6 and 10. [18; 3; 4]

5.2 A complete classification of rank failure

Assume now $\chi \in C^1(I)$. Define

$$D_\chi = 3\,\chi' + 18\,\chi^2 - 2. \tag{5.5}$$

This expression diagnoses a precise obstruction: the dark-radiation column can become a quadratic function of the sound coordinate. It is not a new field equation. [19]

Theorem 5.1 — Exceptional sound histories. The functions $1, J, J^2, E$ are linearly dependent on I if and only if $D_\chi \equiv 0$. In the range $0 < \chi < 1$, all such histories are

$$\chi(s) = \frac{1 - \zeta\, e^{-4s}}{3\left(1 + \zeta\, e^{-4s}\right)}, -\frac{1}{2} < \zeta\, e^{-4s} < 1 \left(s \in I\right), \tag{5.6}$$

for a real constant ζ.

Proof. Put $x = J(s)$, and denote composition with the inverse map by a tilde. Since $d/dx = -Q^{-1}\,d/ds$, repeated differentiation gives

$$\begin{aligned} \tilde{f}_{xxx} &= -Q^{-3}\{f''' + 9(1+\chi)f'' \\ &+ \left[18(1+\chi)^2 + 3\chi'\right]f'\}. \end{aligned} \tag{5.7}$$

Using $E' = -4E$, $E'' = 16E$, and $E''' = -64E$, this reduces to

$$\tilde{E}_{xxx} = \frac{4E}{Q^3} D_\chi, W_s(1, J, J^2, E) = 8\,E\,Q^3 D_\chi. \tag{5.8}$$

Here W_s is the determinant with these four functions as columns and derivative orders zero through three as rows. A zero of the diagnostic at one epoch does not make it vanish on the whole interval.

The first three functions are independent because $J(I)$ contains an interval. Dependence of all four is thus equivalent to $\widetilde{E}$ being a quadratic polynomial. Its third derivative vanishes exactly when $D_\chi \equiv 0$. This uses the polynomial structure directly; it does not assume a general converse from a vanishing Wronskian.

The remaining differential equation is $\chi' = 2/3 - 6\chi^2$. The substitution

$$z = \frac{1-3\chi}{1+3\chi} \Rightarrow z' = -4z \Rightarrow z = \zeta e^{-4s} \tag{5.9}$$

gives (5.6). Solving $0 < \chi < 1$ on this branch gives precisely $-1/2 < \zeta E < 1$. The negative stationary solution $\chi = -1/3$ is excluded; $\zeta = 0$ includes $\chi = 1/3$. This proves completeness. ▫

For $\zeta \neq 0$, direct integration makes the dependence explicit:

$$\begin{aligned} Q &= \frac{\sqrt{1+\zeta E}}{\sqrt{1+\zeta}}, \\ J &= \frac{\sqrt{1+\zeta}}{2\zeta}\left(\sqrt{1+\zeta E} - \sqrt{1+\zeta}\right), \\ E &= 1 + 4J + g_\zeta J^2, \; g_\zeta = \frac{4\zeta}{1+\zeta}. \end{aligned} \tag{5.10}$$

Consequently the coefficient change

$$(\alpha_0, \alpha_1, \alpha_2, d) \mapsto (\alpha_0 - t, \alpha_1 - 4t, \alpha_2 - g_\zeta t, d + t) \tag{5.11}$$

preserves the complete expansion and sound histories. At an interior physical model, sufficiently small changes remain admissible and change d. At a boundary, admissibility must be checked rather than inferred from the null direction alone.

For $\zeta = 0$, $Q = E$ and $J = (E-1)/4$. This familiar radiation-shaped exception permits an affine equation of state; sound one third does not force its constant offset to vanish. For nonzero ζ, the shifted pressure has a modified-Chaplygin form. This connection identifies a known equation-of-state family, not an independently established cosmological matter component. The negative-ζ branch is allowed by the inverse theorem even though its shifted density variable need not be positive. [18; 19]

5.3 Complete histories and finite epochs

Outside the exceptional family, complete exact histories determine at most one coefficient vector. Finite measurements require their own rank condition. At distinct epochs $s_0 < \cdots < s_3$, define $x_i = J(s_i)$ and the matrix A with rows $(1, x_i, x_i^2, E_i)$. The divided-difference identity gives, for an intermediate epoch s_ξ,

$$\det A = \frac{2\,E(s_\xi)\,D_\chi(s_\xi)}{3\,Q(s_\xi)^3} \prod_{0 \le i < j \le 3} (x_j - x_i). \tag{5.12}$$

Indeed, dividing the determinant by the Vandermonde product gives the third divided difference of $\widetilde{E}$, equal to $\widetilde{E}_{xxx}(x_\xi)/3!$. Thus a nowhere-zero diagnostic makes every four distinct epochs nonsingular. If the diagnostic is merely nonzero somewhere, suitable four-epoch designs exist in a neighborhood of that point. A diagnostic that changes sign does not guarantee every chosen design. [19]

For example, constant $\chi = 0.2$ gives $D_\chi = -1.28$; any four distinct epochs have exact coefficient rank four. Constant $\chi = 1/3$ gives rank failure for every design, regardless of the number of repeats. These statements concern exact rank, not achievable accuracy.

The distinction becomes sharp near an exception. If $\chi_\varepsilon = \chi_0 + \varepsilon r$, where χ_0 is exceptional and $r \in C^1(I)$ is fixed, then

$$D_{\chi_\varepsilon} = \varepsilon\,(3r' + 36\,\chi_0 r) + 18\,\varepsilon^2 r^2. \tag{5.13}$$

The definitions of Q, J imply $A_\varepsilon - A_0 = O(|\varepsilon|)$ for fixed epochs. Applying this difference to a unit null vector of A_0 proves $\sigma_{min}(A_\varepsilon) = O(|\varepsilon|)$. This is an upper bound; some perturbations remain exceptional. A small departure may restore uniqueness while leaving the inverse extremely sensitive to noise.

5.4 Differential and weak consistency

Define the differential operator

$$\begin{aligned} L_\chi F &= F''' + 9(1+\chi)\,F'' \\ &\quad + \left[18(1+\chi)^2 + 3\chi'\right] F'. \end{aligned} \tag{5.14}$$

For $F \in C^3(I)$, a fixed constant d admits a four-column representation exactly when

$$L_\chi F + 4\,d\,E\,D_\chi = 0 \text{ throughout } I. \tag{5.15}$$

The proof follows from (5.7): the identity is equivalent to a vanishing third J-derivative of $F - d\,E$, hence to one quadratic on the connected interval. Where

$D_{\chi} \neq 0$, it yields $d = -L_{\chi} F / (4 E D_{\chi})$. Its constancy on only part of the interval does not replace the global condition. [19]

A weak inverse moves derivatives from the measured expansion to a chosen test function. On the increasing interval $J = [J(s_{+}), J(s_{-}))$, choose $\phi \in C^{3}$ with ϕ, ϕ', ϕ'' zero at both endpoints. Set

$$A_{\phi} = \int_{J} \widetilde{F} \phi''' \, dx, B_{\phi} = \int_{J} \widetilde{E} \phi''' \, dx. \tag{5.16}$$

Three integrations by parts annihilate every quadratic, proving $A_{\phi} = d B_{\phi}$. If $B_{\phi} \neq 0$, then

$$d = \frac{A_{\phi}}{B_{\phi}}, |\hat{d} - d| \leq \frac{\varepsilon_{F} \| \phi''' \|_{1}}{|B_{\phi}|} \tag{5.17}$$

when the expansion error is uniformly at most ε_{F}, with exact sound and integration. Moreover,

$$B_{\phi} = -\int_{J} \frac{4 E}{Q^{3}} D_{\chi} \phi \, dx. \tag{5.18}$$

A nonnegative test supported where the diagnostic has one nonzero sign gives a nonzero denominator. All denominators vanish for an exceptional history. Conversely, equality for every compactly supported smooth test means that the distributional third derivative of $\widetilde{F} - d \widetilde{E}$ vanishes, so the continuous function is a quadratic. Agreement for finitely many tests is only a necessary screen.

The integrals can be evaluated in increasing s using the kernel $\phi'''(J(s)) Q(s)$. An alternative is to project E off $span(1, J, J^{2})$ in $L^{2}(I)$: if $R_{E} = E - \Pi E \neq 0$, then $d = \langle R_{E}, F \rangle / \| R_{E} \|^{2}$. Both versions reveal why proximity to an exceptional history magnifies error. Both require full functions or justified integral enclosures. Four expansion samples do not supply a continuous band. Chapter 9 extends the uncertainty accounting when the sound coordinate itself is uncertain. [3; 20]

6 Uncertain Vacuum and Identifiable Bulk Geometry

6.1 Releasing the vacuum intercept

The fixed-vacuum inverse answers a conditional question. If the effective vacuum amplitude is unknown, the expansion coefficients still may be identifiable, but their interpretation changes. Keep the reference $\kappa_0 = g_{ref}/3$, S, a_*, supplied spatial curvature u, and exact sound history fixed. Remove only the known curvature term:

$$\begin{aligned} y = h + u\, e^{-2s} &= \beta_0 + \beta_1 J + \beta_2 J^2 + d\, E, \\ \beta_0 &= b + V + \eta\, V^2, \\ \beta_1 = C(1 + 2\eta V),\ &\beta_2 = \eta\, C^2. \end{aligned} \tag{6.1}$$

Rank four identifies $(\beta_0, \beta_1, \beta_2, d)$, provided the observations are represented by these columns. It does not identify the five quantities (b, V, C, η, d). Counting alone suggests a freedom; the following result finds every allowed representative and shows which geometric quantities survive. [3]

6.2 The complete physical family

Define

$$\Sigma = \beta_1^2 - 4\beta_0\beta_2,\ T(s) = \beta_1 + 2\beta_2 J(s),\ C_{max} = \min_I T(s). \tag{6.2}$$

Theorem 6.1 — Free-vacuum equivalence class. Suppose the identified coefficients satisfy $\beta_2 > 0$, $d \geq 0$, and the common expansion is positive. Every positive-density, positive-tension realization has the unique parameterization

$$\begin{aligned} \eta_C &= \frac{\beta_2}{C^2}, \\ V_C &= \frac{C(\beta_1 - C)}{2\beta_2}, \\ b_C &= \frac{C^2 - \Sigma}{4\beta_2},\ 0 < C < C_{max}. \end{aligned} \tag{6.3}$$

The family is nonempty when $C_{max} > 0$, and is anti-de Sitter precisely when $\Sigma > 0$. A separate prior $b \geq 0$ restricts it to $C^2 \geq \Sigma$.

Proof. Solve $\beta_2 = \eta C^2$ for η, then $\beta_1 = C + 2\beta_2 V / C$ for V, and finally the intercept for b. These steps give (6.3) and prove necessity. Conversely, define

$$v_C(s) = \frac{C\left[T(s) - C\right)}{2\beta_2}, p_C = -v_C + \frac{CQ}{3}. \tag{6.4}$$

Substitution recovers all three coefficients. Since $T' = -2\beta_2 Q < 0$, positivity throughout I is equivalent to $0 < C < C_{max} = T(s_+)$. Differentiation gives $-v_C' = CQ$ and $d\,p_C / d\,v_C = \chi$; conservation and positive enthalpy follow. Finally,

$$\frac{1}{4\eta_C} - b_C = \frac{\Sigma}{4\beta_2}, \tag{6.5}$$

which proves the AdS condition. The zero-density endpoint is excluded. ▫

At a supplied b, this reduces to

$$C = \sqrt{\Sigma + 4\beta_2 b}, \eta = \frac{\beta_2}{\Sigma + 4\beta_2 b}, \tag{6.6}$$

subject to the same physical domain. Thus the earlier fixed-vacuum result remains valid. Replacing its discriminant by Σ while treating b as unknown would give a false unique tension.

6.3 Geometry that survives the freedom

Define the dimensionless curvature combination

$$L = \frac{1}{S\ell^2} = \frac{\Sigma}{4\beta_2} = \frac{\beta_1^2}{4\beta_2} - \beta_0. \tag{6.7}$$

Theorem 6.2 — Invariant bulk metric. Under Theorem 6.1, rank-four identification and $L > 0$ determine

$$\ell = (SL)^{-1/2}, \mu_B = a_*^4 S d, f_k(r) = k + \frac{r^2}{\ell^2} - \frac{\mu_B}{r^2} \tag{6.8}$$

throughout the equivalence class. This follows immediately from (6.5) and the identified d. The bulk metric function is fixed although the matter normalization and split tension vary. On a chosen expanding branch, equal expansion and equal initial scale factor also fix the brane trajectory. [3]

Geometrically, $-L$ is the vertex value of the reconstructed quadratic in J. An affine relabeling $\hat{J} = a_J J + c_J$, $a_J \neq 0$, transforms its coefficients but preserves the vertex value. This is a useful coordinate check. If the vertex lies beyond the visited matter interval, evaluating it remains a model-dependent extrapolation of the quadratic, not a measurement of that unvisited state.

For a planar bulk, $d>0$ gives

$$\frac{r_h}{a_*}=\left(\frac{d}{L}\right)^{1/4}. \tag{6.9}$$

An exterior brane on the whole interval requires $r_h/a_*<e^{s_-}$. For supplied nonzero k, the candidate positive metric root is

$$r_h^2=\frac{\ell^2}{2}\left[-k+\sqrt{k^2+\frac{4\mu_B}{\ell^2}}\right). \tag{6.10}$$

A root and its spacetime interpretation must satisfy the chosen bulk domain. For $k=0, d=0$, zero is not a finite-radius black-hole horizon.

The invariant μ_B has units of length squared. A dimensional bulk mass is proportional to μ_B/G_5, with an additional spatial-volume convention in the planar case. In this family,

$$\lambda_{bare,C}=\frac{SC^2}{2\kappa_0\beta_2}, G_{5,C}=2G_{4,ref}\sqrt{\frac{\eta_C}{S}}, 8\pi G_{4,ref}=g_{ref}. \tag{6.11}$$

Thus invariant geometric mass coefficient does not mean invariant physical mass. Nor does a planar bulk horizon determine the shadow or ringdown of a localized astrophysical black hole.

For smooth exact histories, set $g=y-dE$. The invariant also obeys

$$L=\frac{g'^2}{2\left[g''+3(1+\chi)g'\right)}-g, g''+3(1+\chi)g'=2\beta_2Q^2>0. \tag{6.12}$$

Constancy of this expression is a consistency check within the quadratic model. Its derivative form does not justify twice differentiating noisy distances without an observation and error model.

6.4 Why one density anchor can be insufficient

At an anchored epoch s_a, equation (6.4) gives

$$C^2-T_aC+2\beta_2\nu_a=0, T_a=T\left(s_a\right). \tag{6.13}$$

A quadratic can have two physical roots. An absolute density measurement therefore need not select one representative. By contrast, two densities at distinct epochs give

$$C=\frac{\nu_a-\nu_b}{J_a-J_b}, V=\nu_a-CJ_a, \tag{6.14}$$

and one calibrated enthalpy gives

$$C=\frac{3\left(v_a+p_a\right)}{Q_a}. \tag{6.15}$$

The proofs are respectively the difference of two linear density equations and conservation. Every resulting candidate still must obey the full data and physical constraints. Unknown-gain proxies do not supply these anchors. The densities refer to the stated vacuum convention; the argument does not make two indistinguishable constant vacuum terms separately observable. [3; 4]

An exact worked counterexample makes the two-root issue tangible. Take the planar case $k=u=0$ and choose

$$\begin{gathered} I=\left[-0.5,0.5\right),\chi=0.2, \\ Q=e^{-3.6s},J=\frac{e^{-3.6s}-1}{3.6}, \\ \left(\beta_0,\beta_1,\beta_2,d\right)=\left(9.5632,3.216,0.27,10^{-4}\right). \end{gathered} \tag{6.16}$$

Then $\Sigma=0.0144$, $L=1/75$, and $C_{max}\simeq 3.090794833$. The anchor $v\left(0\right)=1.2$ admits $C=0.216$ and $C=3$.

Table 6.1. Two representatives sharing one density anchor.

Quantity	First representative	Second representative
C	0.216	3.000
$v\left(0\right)$	1.200	1.200
η	5.787037037	0.030000000
b	0.029866667	8.320000000
L	0.013333333	0.013333333
r_h/a_*	0.294283096	0.294283096

Both have positive density, enthalpy, expansion and vacuum amplitude. Their common horizon lies below the earliest brane radius $e^{-0.5}\simeq 0.606530660$. Their split tensions differ by $\left(3/0.216\right)^2\simeq 192.90$. This is an analytic construction, with rounded table entries, illustrating identical restricted information rather than two observed universes. [3]

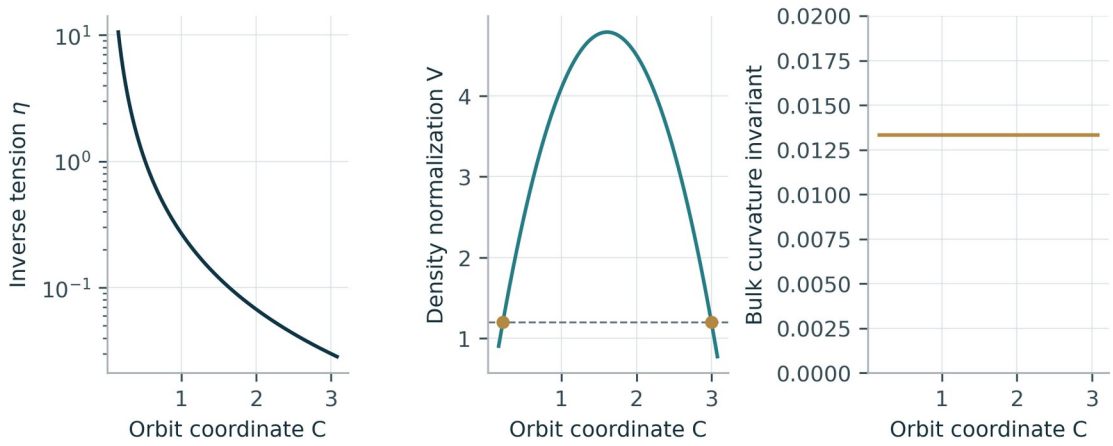

Figure 6.1. The equivalence class of Equation 6.16. The inverse tension and density normalization vary with C, while the bulk curvature combination remains 1/75. The two marked representatives share the density anchor v(0)=1.2. The figure uses the planar example and its stated physical interval.

6.5 From a reconstruction orbit to an action convention

The physical meaning of this family requires more care than its algebra. Relative to one representative, let $\alpha = C_{new}/C > 0$. Equations (6.3)–(6.4) give

$$\begin{aligned} v_\alpha &= \alpha\, v + \frac{\alpha(1-\alpha)}{2\eta}, \\ p_\alpha &= \alpha\, p - \frac{\alpha(1-\alpha)}{2\eta}, \\ \eta_\alpha &= \frac{\eta}{\alpha^2}, b_\alpha = b + \frac{\alpha^2-1}{4\eta}, \\ \lambda_{bare,\alpha} &= \alpha^2 \lambda_{bare}, G_{5,\alpha} = \frac{G_5}{\alpha}. \end{aligned} \tag{6.17}$$

If the microscopic matter action may change, this transformation can multiply the entire classical action by α. For the scalar realization $F(X) - U_0$, with $8\pi G_5 = M_5^{-3}$, one implementation is

$$\begin{aligned} M_{5,\alpha}^3 &= \alpha\, M_5^3, F_\alpha = \alpha\, F, \\ U_{0,\alpha} &= \alpha\, U_0 - (\alpha^2 - \alpha)\lambda_{bare}, \\ \tau_{vac} &= \lambda_{bare} + U_0, \tau_{vac,\alpha} = \alpha\, \tau_{vac}. \end{aligned} \tag{6.18}$$

With geometric bulk cosmological constant fixed, the bulk, boundary and brane actions scale together. Classical stationary configurations are unchanged. Scaling a whole action at fixed $\hbar$ is not generally a quantum equivalence, and the full transformation is unavailable if other matter sectors have independently fixed absolute normalizations. [3; 4]

After subtracting the declared constant vacuum, the coefficient of linear excitation density is different from the reference split coefficient:

$$g_{ref} = g_{split} = \frac{\lambda_{bare}}{6\,M_5^6},$$
$$g_{bg} = \frac{\tau_{vac}}{6\,M_5^6}, \tag{6.19}$$
$$\left(g_{split}\right)_\alpha = g_{split}, \left(g_{bg}\right)_\alpha = \frac{g_{bg}}{\alpha}.$$

An exact nonzero absolute determination of g_{bg} can therefore select at most one member of this orbit. Identifying it with locally measured $8\pi G_N$ needs low-energy localization, fixed stress standards and negligible extra scalar or bulk contributions. Chapter 10 supplies the action-level account; a local source matching calculation has not been completed for every cosmological scalar state. This conditional route does not contradict the earlier inverse at fixed g_{ref}.

There is also a separate, exact parameter redundancy:

$$\lambda_{bare} \mapsto \lambda_{bare} + \Delta, U_0 \mapsto U_0 - \Delta. \tag{6.20}$$

Only their sum occurs in the action. No observable of that identical action can distinguish the two labels without additional structure. An absolute gravitational calibration removes the stated scaling orbit under its matching conditions; it does not remove this vacuum-split redundancy. The distinction is already rooted in the brane stress decomposition and becomes decisive when a reconstructed history is promoted to a dynamical model. [7; 4]

6.6 The limits of the invariant statement

The Riccati exceptions of Chapter 5 survive the release of *b*. At $\chi = 1/3$, changing the coefficients by $t(-1, -4, 0, 1)$ preserves complete histories while changing *d* and generally L. Bulk invariance is therefore conditional on coefficient identification, not guaranteed by any expansion-plus-sound pair. [19; 3]

Outside that exception, finite measurements still require informative epochs, adequate precision and a validated sound model. Additional density differences must give one common C, and calibrated enthalpies must make $3(v+p)/Q$ constant. These are operational restrictions on the declared model. Chapter 9 propagates uncertainty through the joint sets so that neither a spurious unique tension nor an artificially precise bulk geometry is created by substituting fitted sound values as though they were exact.

7 Sound Windows and Spectral Recovery

7.1 A logarithmic modulation is a hypothesis

The conservation coordinate of Chapter 5 makes prescribed sound histories tractable. It does not select their functional form. One candidate is a sinusoid in logarithmic scale factor,

$$\chi(s)=\chi_0+A\cos(\omega s+\delta),0<|A|<min(\chi_0,1-\chi_0). \tag{7.1}$$

The inequality supplies a strict sound margin. Choosing $\omega=2\pi/\log\varphi$, where $\varphi=(1+\sqrt{5})/2$, inserts a golden-ratio period into the matter model. Neither the Einstein equations nor a drawing of an expanding spiral derives that choice. A monotone scale factor can be displayed on a logarithmic spiral with a freely assigned angular coordinate. Such a display changes the representation of a history rather than its dynamics. [21]

For (7.1), conservation gives

$$\begin{aligned} Q(s)&=\exp\{-3(1+\chi_0)s\\ &-3A/\omega[\sin(\omega s+\delta)-\sin\delta)]\},\\ J(s)&=-\int_0^s Q(t)\,dt,v=V+CJ. \end{aligned} \tag{7.2}$$

Every nonconstant finite sum of distinct positive-frequency harmonics is outside the complete-history exception of Chapter 5. Indeed, let ω_K be its highest frequency with amplitude $A_K\neq0$. The diagnostic $3\chi'+18\chi^2-2$ contains a harmonic at $2\omega_K$ with amplitude $9A_K^2$. No derivative term or product of lower frequencies can cancel it. Consequently $1,J,J^2,e^{-4s}$ are functionally independent. This proof works for arbitrary distinct positive frequencies; it grants no special dynamical status to the golden choice. [22]

Functional independence does not guarantee a well-conditioned sampled matrix. For example, when the mean sound is $1/3$, expansion epochs taken at one repeated modulation phase can make the sampled dark-radiation column affine in J. The sampled rank then falls even though the continuous functions are independent. More observations at the same aliased phases do not repair that design. [21; 23]

7.2 What a sound window measures

Assume an observation is a uniform average of χ over a width $w>0$ centered at c, both measured in $s=\log(a/a_*)$. For a harmonic its response is

$$W_w\chi(c)=\chi_0+A\,m_w(\omega)\cos(\omega c+\delta),$$
$$m_w(\omega)=sinc(\omega w/2),\ sinc\,x=\frac{\sin x}{x}. \tag{7.3}$$

These are stipulated effective-fluid sound observations. The operator averages χ itself in logarithmic scale factor. Averaging its square root, using cosmic time, or interpreting it as the baryon acoustic sound horizon would change the measurement. An astronomical application needs a forward model connecting perturbation measurements to this operator and its joint errors. The results below begin after that observation model has been supplied. [15; 21]

Two widths can recover an unknown single frequency without fitting the gravitational sector. Measure widths Δ and 2Δ at common centers, and remove each vector's constant component with $P_N=I-11^T/N$. Let $U=P_N y^{(1)}$ and $Z=P_N y^{(2)}$. The identity $sinc(2x)=\cos x\,sinc\,x$ gives

$$Z=r\,U,\ r=\frac{U^TZ}{U^TU}=\cos(\omega\Delta/2),\ \omega=\frac{2}{\Delta}\arccos r. \tag{7.4}$$

This inverse requires $U\neq 0$ and a declared frequency branch inside $(0,2\pi/\Delta)$. Frequency recovery requires calibrated relative gains and window widths. Centering can protect the frequency ratio against constant channel offsets; it does not remove an unknown relative multiplicative gain. Once the frequency is known, three distinct phases permit recovery of the physical sound baseline χ_0 and both quadratures, provided the absolute gain and zero point are also calibrated. [23]

The inverse also displays its instability. With deterministic centered errors of norms at most ϵ_1,ϵ_2, an observed ratio formed from $\widetilde{U},\widetilde{Z}$ obeys

$$|\hat{r}-r|\le\frac{\epsilon_1+\epsilon_2}{\|\widetilde{U}\|},\ \widetilde{U}\neq 0. \tag{7.5}$$

Intersect this enclosure with the cosine image of the allowed branch before applying the decreasing arccosine. An empty intersection indicates incompatibility with the assumed signal and error bounds. A small centered response yields a broad enclosure even if the algebraic ratio can be computed. Near a branch endpoint, arccosine further amplifies error. [23]

7.3 Multiple modes need additional structure

A second oscillation changes the paired-width ratio: it is generally no longer the cosine of either frequency. Nor is it necessarily a positive weighted average of their cosines; that interpretation requires orthogonal centered mode responses. Omitting a mode can bias the inferred gravitational coefficients while all final coefficients remain physically admissible. [22]

Three commensurate widths provide an exact two-mode construction. Assume $0<\omega_1<\omega_2<2\pi/\Delta$, so both frequencies lie on the same injective cosine branch. Set $c_j=\cos(\omega_j\Delta/2)$, with $c_1\neq c_2$, and let z_j be their centered first-width responses. For widths $n\Delta$, define $H_n=n\,P_N\,y^{(n)}$. The Chebyshev identity gives

$$\begin{aligned} H_n&=\sum_{j=1}^{2} U_{n-1}(c_j)\,z_j,\\ H_3&=2t\,H_2-(4\,p_f+1)\,H_1,\\ t&=c_1+c_2,\ p_f=c_1c_2. \end{aligned} \tag{7.6}$$

If z_1, z_2 are independent, the recurrence determines t, p_f; the roots of $x^2-t\,x+p_f$ give the two cosines. A single arccosine branch removes the branch ambiguity, and a separate rank-five condition on the uncentered design recovers the constant and four quadratures. Two real roots from noisy data alone do not establish two physical modes. Close roots, nearly dependent response vectors and an omitted third mode all undermine interpretation. [22]

One synthetic example in the source makes the risk concrete:

Table 7.1. Source-reported bias from omitting a sound mode.

Model used to fit two-mode truth	Recovered η	Recovered d
Generating values	0.030000	0.150000
Misspecified single mode	0.027485	0.116383

The single-mode coefficients pass the stated physical screen. Their disagreement is therefore model error, not a violation that a positivity filter necessarily detects. These are source-reported synthetic calculations under supplied b, u and scales; they are neither astronomical estimates nor new numerical reruns for this chapter. [22, Table 3]

7.4 Exact blindness and stable design

An integer number of periods within every window erases a harmonic exactly. For widths $w_j=j\Delta$, the frequency $2\pi/\Delta$ has $m_{w_j}=0$ for every j. An irrational ratio

between two widths removes such common zeros. For complete center functions on an open interval, this suffices for injectivity on finite trigonometric sums: linear independence of distinct harmonics forces any invisible harmonic to vanish in every channel. [24]

Two qualifications are essential. First, finitely many measurements cannot identify an unrestricted number of modes. If the number of unknown real coefficients exceeds the number of observations, a linear null direction exists; a small perturbation in that direction can preserve a strict sound margin. Second, absence of exact zeros says nothing by itself about noise amplification. Define

$$T_w(\omega)=\left[\sum_j m_{w_j}(\omega)^2\right]^{1/2},\gamma_\Omega=\min_{0\le\omega\le\Omega} T_w(\omega). \tag{7.7}$$

The positive floor on a declared compact band quantifies transfer visibility. Every fixed collection of positive widths still has $T_w(\omega)=O\left(|\omega)^{-1}\right)$ as frequency grows, so no all-frequency positive floor exists. With widths $(\Delta,2\Delta,\varphi\Delta)$, the Fibonacci frequencies $\omega_n=2\pi F_n/\Delta$ satisfy

$$T_w(\omega_n)=\frac{\sin\left(\pi\varphi^{-n}\right)}{\pi\varphi F_n},F_n^2T_w(\omega_n)\to\frac{1}{\sqrt{5}\varphi}. \tag{7.8}$$

Golden widths remove common zeros but retain increasingly weak responses. This is a property of the chosen averaging design, not evidence for Fibonacci dynamics in the Universe. [24]

A usable numerical design criterion needs an enclosure between grid points. Since $|sinc'x)\le 1/2$, the transfer norm is Lipschitz with constant $L_\omega=\|w\|_2/4$. A frequency grid of maximum spacing h_ω, including both band endpoints, supplies

$$max\{0,g_{grid}-L_\omega h_\omega/2\}\le\gamma_\Omega\le g_{grid}, \tag{7.9}$$

where g_{grid} is the smallest grid response. Width uncertainty further reduces the guaranteed floor by at most $\Omega\|\delta w\|_2/4$. Rounding error requires separate control if a formally certified computational bound is claimed. [24]

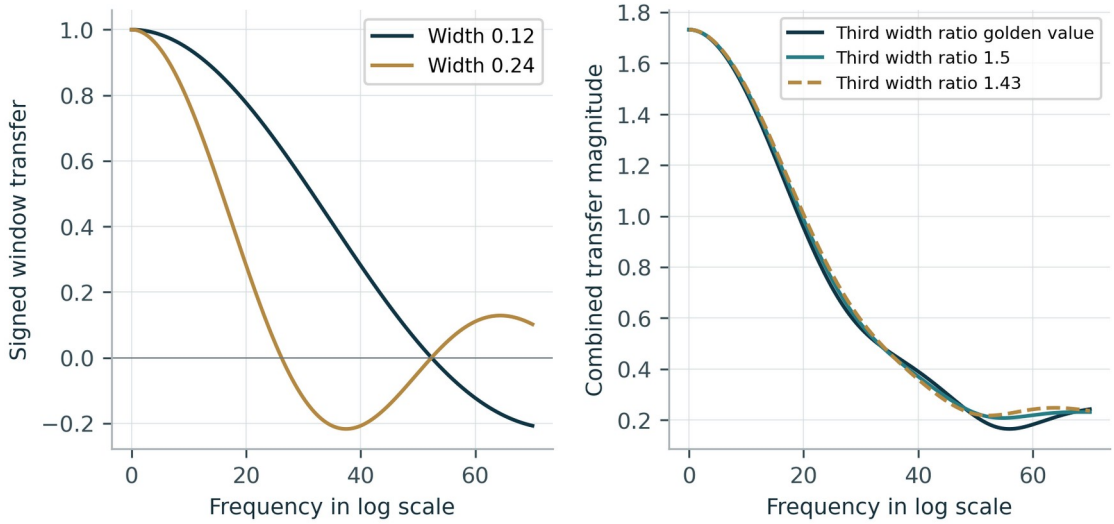

Figure 7.1. Uniform sound-window transfer on the declared frequency band. Left: the signed responses of widths 0.12 and 0.24. Right: their combined magnitude with a third width of ratio 1.5, the golden value, or 1.43. A finite-band design comparison does not establish an all-frequency stability guarantee.

For the source's declared band $[0, 70)$ and $\Delta = 0.12$, the grid lower bounds from (7.9) are approximately 0.20709 for width ratio 1.5, 0.16421 for φ, and 0.21684 for the selected ratio 1.43, each used alongside $\Delta, 2\Delta$. A rational design can therefore outperform a golden design on a finite band even though it has a common zero elsewhere. The criterion and frequency band determine the comparison. Finite-center conditioning, covariance and the final physical target must also enter an actual design decision. [24, Table 2]

7.5 When expansion adds independent information

For a supplied sound parameter vector, expansion retains the linear four-column form of Chapter 5. That physical inverse assumes fixed b; its coefficient precision cannot be transferred unchanged to the free-vacuum tension family of Chapter 6.

The statistical information statements in Sections 7.5 and 7.6 assume Gaussian errors with fixed known covariance and independent sound and expansion blocks. The Jacobian rank identity itself is deterministic and does not require that probability law. In the corresponding whitened coordinates, let $\overline{A}$ be the expansion matrix, X the sound Jacobian and D the expansion derivatives with respect to sound parameters. The combined Jacobian is

$$G = \begin{pmatrix} X & 0 \\ D & \overline{A} \end{pmatrix}, P_A^{\perp} = I - \overline{A}\,\overline{A}^{+}. \tag{7.10}$$

Proposition 7.1 — Independent expansion information. If $\overline{A}$ has rank four, the joint Jacobian has full column rank exactly when

$$ker\, X \cap ker\left(P_A^{\perp} D\right) = \{0\}. \tag{7.11}$$

Proof. A null perturbation must satisfy $X\,\delta\theta_s = 0$ and $D\,\delta\theta_s + \overline{A}\,\delta\alpha = 0$. The second equation has a solution precisely when its sound perturbation lies in the gravity-column span, equivalently when $P_A^{\perp} D\,\delta\theta_s = 0$. Full rank of $\overline{A}$ then eliminates any remaining pure gravity null vector. ▫ [25]

This explains why expansion can help with a sound-blind mode. It also explains why four expansion epochs are insufficient to guarantee that help: an invertible square A can absorb small sound changes by changing its four coefficients. At an interior physical model, sufficiently small perturbations preserve the strict physical inequalities, so this coefficient freedom gives nearby physical alternatives. More generally, if sound leaves r_0 first-order directions undetermined, at least r_0 expansion residual dimensions beyond the four gravity columns are necessary. Several hidden directions must remain jointly independent after projection; checking each column separately is insufficient.

Even when rank is restored, information can be small. For an infinitesimal high-frequency mode added to a fixed smooth base history, integration by parts gives $\left\| P_A^{\perp} D_{\Omega} \right\| = O\left(\Omega^{-2}\right)$ at fixed epochs and noise. Leading terms lie in the gravity-column span and disappear on projection. At a sound common zero, the resulting local amplitude information can decay as Ω^{-4}. This is a local derivative statement around the specified base, not a bound on every finite nonlinear modification. [25]

7.6 Unknown frequency and weak signals

An unknown upper frequency introduces another distinction. Let G_v contain the nuisance derivatives, including its two amplitude quadratures, and let h_{Ω} be the frequency derivative of the full whitened mean. If G_v has full column rank, the effective local frequency information is

$$I_{\Omega} = \left\| \left(I - G_v G_v^{+}\right) h_{\Omega} \right\|^2. \tag{7.12}$$

Positive I_{Ω} gives a regular local inverse direction. The associated local standard-deviation scale is $I_{\Omega}^{-1/2}$; it is not a guarantee that a nonlinear search finds the correct frequency globally. At a window zero, the amplitude response may vanish while its frequency derivative remains nonzero. At zero physical amplitude, however, the entire mean is independent of that frequency. [26]

On a bounded interval, the derivative of a harmonic with respect to frequency is proportional to its amplitude magnitude. Here $A \geq 0$ denotes the Euclidean norm of the upper harmonic's two quadrature coefficients. The corresponding conservation and expansion derivatives obey the same amplitude scaling. Consequently

$$\| h_{\Omega} \| \leq A\, C_{\Omega}, I_{\Omega} \leq A^{2} C_{\Omega}^{2}, \tag{7.13}$$

for a constant determined by the fixed physical and observation bounds. Nearby frequency models therefore approach the same probability law as $A \to 0$. Reporting a precise frequency when the component itself is absent cannot be justified by optimization alone. Separated bands and nonzero signal assumptions should be visible whenever regular frequency errors are quoted. [26]

Table 7.2. Distinct questions in spectral reconstruction.

Question	Required control	What it does not establish
Is a harmonic visible?	Window transfer and finite design rank	Adequate precision
Can sound and expansion recover local parameters?	Joint projected Jacobian rank	A global unique solution
Is an added frequency estimable?	Nonvanishing effective information	Detection of an added component
Is a selected mode significant?	Calibration of the actual search and null	A cosmological interpretation

These distinctions determine the statistical problem. A frequency chosen because it gives the largest apparent improvement is no longer a prespecified frequency. Its reference distribution must account for that search, the nuisance family and the covariance assumptions. Chapter 8 develops those controls before any spectral claim is carried into physical reconstruction.

8 Selection and Covariance in Statistical Validation

8.1 State the experiment before stating significance

A fitted oscillation, a rejected null hypothesis and a confidence interval answer different questions. Their guarantees also depend on different observation models. Series II moves from a nonlinear joint sound–expansion search to sound-only validation, then from known covariance to independent complete repeats with unknown covariance. These transitions add useful results by changing assumptions or statistical targets. They cannot be compressed into one unrestricted claim that a detected frequency has been validated. [27; 28; 29; 30]

In the known-covariance experiment, a specified fixed whitening gives $z=\left(z_s, z_F\right) \sim N_N\left(g, I_N\right)$. The source design has 51 sound means and 41 expansion coordinates, with known sound covariance and independent known-variance expansion errors. Other designs require their own matrices. The theoretical guarantees do not attach to the number 92; they attach to the stated mean, covariance and independence conditions.

Table 8.1. Statistical experiments and their validation targets.

Procedure	Observation requirement	Statistical target
Calibrated nonlinear joint search	Known Gaussian covariance; declared null references	Improvement produced by the specified fitting algorithm
Selected sound validation	Known Gaussian covariance; independent information split	A component in a selected sound subspace
Validation with uncertain primary frequency	Previous conditions and fixed bounds on the primary family	An added projected component after bounded primary leakage
Validation with unknown covariance	Independent, identically distributed complete Gaussian repeats	The same selected component with covariance uncertainty included

The sound-only null permits an arbitrary expansion mean. Expansion may guide selection without supplying the held-out validation statistic. This enlarged null covers the inherited physical nulls but changes what is being tested. Its sound guarantee can survive a change to the gravity intercept when the sound and noise assumptions remain valid. A joint search calibrated for a fixed-vacuum physical null requires renewed calibration if that null or its score changes. [28; 30; 3]

8.2 Calibrating a nonlinear search algorithm

Let a fully specified fitting procedure return residual sums R_0, R_1 for the null and alternative. Define

$$T_A(z) = max\{0, R_0(z) - R_1(z)\}. \tag{8.1}$$

The procedure includes frequency bands, initializations, stopping rules, parameter screens and a rule for failures. Without certified global minima, this is an algorithm-defined improvement score. A local optimizer can nevertheless yield a valid test if its complete score distribution is calibrated under the null. It does not yield a certificate excluding every unvisited parameter value. [27]

For a specified null $g_0(v)$, independently generate B observations from the same Gaussian law and apply the identical procedure. The Monte Carlo p-value is

$$p_B(v; z) = \frac{1 + \sum_{b=1}^{B} 1\{T_A(z_b) \geq T_A(z)\}}{B+1}. \tag{8.2}$$

Proposition 8.1 — Specified-null calibration. Under that null,

$$Pr\{p_B \leq \alpha\} \leq \frac{\lfloor \alpha(B+1) \rfloor}{B+1} \leq \alpha. \tag{8.3}$$

Proof. The observation and calibration scores are exchangeable. Without ties, the observed descending rank is uniform on $1, \ldots, B+1$. Counting ties with “greater than or equal” only increases the reported p-value. Counting the ranks that permit rejection proves the bound. ▫ [27]

A shorter search on simulated data, a frequency band chosen from the observation but held fixed in simulation, or selective removal of failed observed fits breaks that exchangeability argument. Increasing the number of simulations cannot repair a mismatch in the procedure.

Unknown null parameters need additional treatment. A maximum $p_V = max_{v \in V} p_B(v; z)$ controls every member of a fixed finite reference set V, since it dominates the p-value at the true reference. No multiplicity factor over these references is needed. A supremum over the complete null family would similarly be valid, but an attained local maximum is a lower bound on that supremum and cannot certify rejection. A finite reference grid does not establish continuum coverage merely by being dense. [27]

One explicit extension uses distances between whitened Gaussian means. Their total variation distance is

$$v_{TV}(r)=2\Phi(r/2)-1,\ p_{cover}=min\{1,\ p_V+v_{TV}(r)\}. \tag{8.4}$$

If every allowed null mean lies within distance r of a reference, this correction controls that covered family. For $0<\alpha<1$, if $\alpha\le v_{TV}(r)$, rejection is impossible for the finite Monte Carlo p-value. Otherwise, rejection under the nearby reference has probability at most $\alpha-v_{TV}(r)$; changing only the observation law increases it by at most the total variation distance. The simulations remain independent with their original law. Coverage of the mean-space neighborhood is an assumption to establish, not a consequence of plotting reference fits. [27]

A fitted-null bootstrap uses $\hat{v}(z)$ to generate its calibration law. The observed vector is then not an independent draw from a null fixed at that estimate. Such a bootstrap may have validity under additional conditions, but the rank proof above does not establish it. The $+1$ correction addresses simulation rank, not estimated-null dependence.

8.3 Independent validation with known covariance

Suppose the primary frequency is supplied. Let X_0 contain the whitened constant and two primary quadratures, and let

$$P=I-X_0X_0^{+}. \tag{8.5}$$

Under the enlarged sound null $g_s\in col(X_0)$, the projected mean vanishes exactly. A prespecified finite-frequency scan of projected noise can therefore be calibrated by the same exchangeability argument without estimating physical nuisance parameters. Selection using the same residual still has to be included in that scan; a fixed-frequency chi-squared reference does not calibrate a maximum over frequencies. [28]

An alternative reserves independent information for validation. For a fixed fraction $0<f<1$ and independent $\xi\sim N_N(0,I_N)$, set

$$\begin{aligned} z^{tr}&=z+\sqrt{(1-f)/f}\,\xi,\\ z^{va}&=z-\sqrt{f/(1-f)}\,\xi. \end{aligned} \tag{8.6}$$

Their cross covariance is zero; joint Gaussianity makes them independent. Their covariances are I/f and $I/(1-f)$, and $z=f\,z^{tr}+(1-f)\,z^{va}$. Thus randomization partitions information without creating another observation. The selector may inspect training sound and expansion, but not validation, the original vector or the

randomization separately. Either of the latter, together with training, can reveal held-out information. [28]

Let training choose an orthonormal basis U in the primary-residualized upper-mode space, of rank $r \le 2$. Conditional on training,

$$T_{va} = (1-f) \left\| U^T P z_s^{va} \right\|^2 \sim \chi_r^2 \tag{8.7}$$

under the null. The proof is immediate: the projected validation vector has zero mean and identity covariance after multiplication by $\sqrt{1-f}$. Rank zero abstains. An inaccurate training fit reduces usefulness or power; it does not change this conditional Gaussian law.

For a training-chosen unit direction q, the corresponding interval targets $\psi = q^T P g_s$, the total projected sound mean. It does not cover an unprojected amplitude, true frequency, tension or bulk mass. Furthermore, if the upper harmonic is exactly erased by every sound window, the validation mean is unchanged under its presence. Expansion may identify a promising frequency during training while sound validation remains at null-level power. That statement concerns this sound-only route, not every joint test. [28]

Known covariance is essential to the artificial split. If the actual covariance of z is $I + E$, the cross covariance of the two components is E. Nominal whitening with an estimated or incorrect covariance therefore does not inherit independence.

8.4 A primary frequency interval leaves bounded leakage

If the primary frequency lies in an externally supplied interval, no fixed linear contrast can remove every primary harmonic in that interval while retaining other harmonic means. To see why, associate a finite-window contrast with its compactly supported kernel $k_q(t)$. Its Fourier transform

$$F_q(v) = \int k_q(t) e^{ivt} dt \tag{8.8}$$

is entire. Removing both quadratures on a nonempty open frequency interval makes this function vanish there, hence everywhere by the identity theorem. This obstructs exact linear removal; it does not rule out nonlinear nuisance inference or controlled approximate removal. [29]

A practical construction projects the constant and the first K frequency derivatives at a nominal ω_c. The primary-only sound mean is $g_{s,0} = \chi_0 c + X_\omega a$, where c is the constant column and X_ω contains the two harmonic columns in the whitened sound coordinates. Restrict $|\omega - \omega_c| \le h_\omega$, $0 \le \chi_0 \le 1$, and $\| a \| \le A_{max}$, while leaving the

expansion mean unrestricted. For any selected orthonormal U, define $d_j = \| U^T X_{\omega_c}^{(j)} \|_{op}$ and a uniform derivative bound M_{K+1}. The window kernels provide such a bound through their absolute moments. Taylor's theorem gives

$$\begin{gathered} R_U \le \sum_{j=0}^{K} \frac{h_\omega^j d_j}{j!} + \frac{h_\omega^{K+1} M_{K+1}}{(K+1)!}, \\ R_U = \sup_{\omega} \| U^T X_\omega \|_{op}, \\ B_U = \| U^T c \| + A_{max} R_U, \| U^T g_{s,0} \| \le B_U. \end{gathered} \tag{8.9}$$

Here c is the transformed constant column and $g_{s,0}$ the primary-only mean. In implementation, replace R_U by its computable upper bound. A grid can tighten it only with a justified between-grid derivative correction. Small residual derivative columns must be retained when exact annihilation is not achieved numerically. [29]

Training may choose the frequency and projection order. Under known covariance, $Z = \sqrt{1-f}\, U^T z_s^{va}$ has identity covariance and null mean norm at most $b_U = \sqrt{1-f}\, B_U$. Its squared norm therefore has noncentrality no larger than b_U^2. Comparing $\| Z \|^2$ with the upper tail of $\chi_r^2(b_U^2)$ is conservative because that family is stochastically increasing in noncentrality. This yields conditional control over the bounded primary family without assuming that a fitted primary projector exactly removes the truth. [29]

The interval, amplitude bound and covariance must be justified independently of the tested observation. The source's amplitude bound of one half follows for a globally causal single sinusoid; it does not follow from causality only on a finite observed segment. With primary leakage retained, the confidence target is now the projected added component, rather than the total projected mean of the previous section.

8.5 Unknown covariance requires a different observation model

Let L_0 be a fixed invertible nominal transformation of the sound block. The complete vectors Y_j below are already expressed in those coordinates, with $c = L_0 1$ and $X_v = L_0 W_v$, where W_v contains the untransformed window responses of Chapter 7. The unknown covariance Σ refers to these same coordinates; L_0 need not whiten it. The unknown-covariance construction assumes complete repeats

$$Y_j \overset{iid}{\sim} N_N(g, \Sigma), \Sigma \succ 0 \text{ unknown}. \tag{8.10}$$

Within a repeat, sound and expansion coordinates may have arbitrary correlations. Across repeats, independence, common mean and common covariance are essential. Distinct redshift bins, overlapping windows, or repeated readouts sharing calibration error are not automatically independent complete vectors. A common cosmic-variance component also violates this contract. [30]

Disjoint repeat groups provide independent training and validation without knowing Σ. Training chooses U and a valid primary radius B_U using the same transformed window columns. In n validation repeats let $V_i = U^T Y_{i,s}$, with sample mean $\overline{V}$ and usual unbiased covariance S_V. For $n > r > 0$, form

$$D = n \min_{\|m\| \le B_U} (\overline{V} - m)^T S_V^{-1} (\overline{V} - m). \tag{8.11}$$

Theorem 8.2 — Hotelling domination. Write $a_{n,r} = (n-r)/[r(n-1)]$. Rejecting when $a_{n,r} D$ exceeds the $(1-\alpha)$ quantile of $F_{r,n-r}$ has conditional null probability at most α.

Proof. The true projected primary mean m_0 belongs to the nuisance ball. Hence, pointwise,

$$\begin{gathered} D \le n(\overline{V} - m_0)^T S_V^{-1} (\overline{V} - m_0) = T_0^2, \\ a_{n,r} T_0^2 \sim F_{r,n-r}. \end{gathered} \tag{8.12}$$

The second line is the classical Gaussian one-sample Hotelling law, conditional on training. The rejection event for D is contained in the corresponding level-α event for T_0^2. Integrating over training preserves the bound. ▫ [30]

The minimized statistic $a_{n,r} D$ does not generally have an exact F distribution. The reference dominates it. The proof integrates over the validation mean and covariance estimate together; it is not conditional on the realized S_V. Only an at-most-two-dimensional covariance is estimated, so the requirement is $n > r$, not $n > N$. Rank zero or $n \le r$ requires abstention for this construction.

The ball minimization is a convex problem. If $\overline{V}$ lies in the ball, $D = 0$. Otherwise, diagonalize $S_V = O\,\mathrm{diag}(s_j) O^T$ and write $z = O^T \overline{V}$. For $B_U > 0$, its unique solution is

$$m_* = O\left(\frac{z_j}{1 + \lambda_{opt} s_j} \right)_j, \quad \sum_j \frac{z_j^2}{(1 + \lambda_{opt} s_j)^2} = B_U^2. \tag{8.13}$$

The multiplier $\lambda_{opt} > 0$ is a numerical optimization variable, unrelated to brane tension. The left side decreases continuously to zero, so a bracketed scalar solve determines it. At $B_U = 0$, the only possible nuisance mean is zero. [30]

For component confidence statements, assume the alternative mean decomposes as $g_s = g_{s,0} + g_{s,add}$, with the primary contribution satisfying the same family bounds. The target is $U^T g_{s,add}$. Subtracting a proposed added component before applying (8.11) gives a confidence region: the Minkowski sum of $\overline{V}$, the nuisance ball and the centered Hotelling ellipsoid. It simultaneously covers all directional contrasts within the training-selected subspace. A direction chosen after validation may be reported from that whole region. A smaller scalar Student interval instead requires the direction to have been chosen using training alone. [30]

For that scalar route, let the primary contrast lie in $\left[l_q, u_q\right)$, and let $\overline{Z}, s_q$ be the validation mean and standard deviation. The added contrast has the interval

$$\left[\overline{Z} - u_q - t_* s_q/\sqrt{n}, \overline{Z} - l_q + t_* s_q/\sqrt{n}\right), t_* = t_{n-1,1-\alpha/2}. \tag{8.14}$$

It follows by applying the centered Student law to the total mean and subtracting every allowed primary contribution. Neither this interval nor the projected region identifies a physical frequency or the complete matter history. Rejecting whenever either a directional test or the ball test is significant also needs multiplicity control.

8.6 What numerical validation can establish

The cost of covariance estimation is visible even without frequency selection or primary leakage. With $r = 2$, $B_U = 0$ and $n > 2$, inserting sample covariance into a chi-squared reference gives null rejection probability

$$Pr\{T_0^2 > \chi^2_{2,0.95}\} = \left(1 + \frac{\chi^2_{2,0.95}}{n-1}\right)^{-(n-2)/2}. \tag{8.15}$$

It is approximately $33.4\,\%$, $15.6\,\%$ and $8.5\,\%$ at $n = 4, 8, 20$, respectively, instead of $5\,\%$. This classical finite-sample effect explains why an estimated covariance cannot simply replace a known covariance while leaving the reference law unchanged. [30]

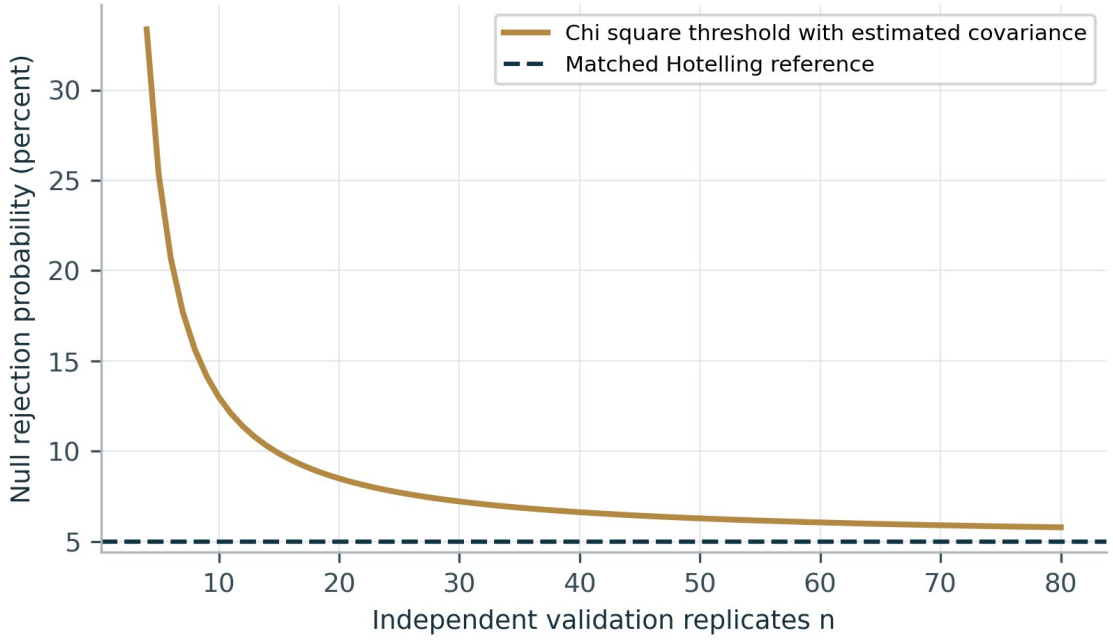

Figure 8.1. Exact null rejection probabilities for a rank-two Gaussian validation mean. A chi-squared threshold with an estimated covariance exceeds its nominal five-percent size. The matched Hotelling reference retains five-percent size in the same independent-replicate experiment. This is an analytical calculation, not a new simulation ensemble.

The source simulations also distinguish formula validity from implementation evidence. Paper VII reports naive nonlinear-search rejection rates of 25.2 %–30.4 % in its chosen null cells; these rates belong to that algorithm and experiment. Paper X retains a lowest corrected directional coverage of 91.5 % in its small main cells. A later 20,000-batch diagnostic for one fixed training choice gives 94.93 % directional and 95.04 % region coverage. The latter check does not replace the original shortfall or establish uniform empirical coverage. These figures are source-reported synthetic results, not newly reproduced runs or observations of the Universe. [27; 30]

8.7 Allocate the error budget across the scientific claim

A validated sound contrast is not a uniform band for $\chi(s)$. It cannot directly replace the upstream function uncertainty required to reconstruct J, the expansion matrix and the entire physical identified set. Chapter 9 builds that separate route. Likewise, a local frequency error bar should not be relabeled as evidence for an added mode when its amplitude can vanish. [26; 20]

Suppose a test controls false rejection at α_T, and a compatible physical confidence construction controls noncoverage at α_C, under common assumptions. Then

$$Pr\{\text{false rejection or physical noncoverage}\} \le \alpha_T + \alpha_C. \tag{8.16}$$

No independence is required. Two separate five-percent guarantees yield a ten-percent union bound, not an automatic joint ninety-five-percent statement. Moreover, unconditional physical coverage need not remain ninety-five percent after restricting reports to datasets with a discovery. Conditional reporting needs its own coverage argument. [6]

The defensible sequence is therefore explicit: state the physical family and observation model, specify what selection may inspect, calibrate the resulting statistic, name its exact confidence target, and propagate a justified common uncertainty event to the physical quantities. Stronger claims require stronger premises at the step where they enter, rather than a more impressive significance number at the end.

9 Confidence Sets for Physical Parameters

9.1 Retaining everything the observations permit

A fitted sound curve is an uncertain input to the gravitational inverse. Treating it as exact can make a coefficient estimate look precise while ignoring the uncertainty that changes its design matrix. A second issue is structural: different physical models may produce the same observation mean. A valid confidence procedure should retain that entire identified set on one coverage event. More repeated measurements can reduce statistical uncertainty without resolving an exact equivalence. [20; 6]

The construction below has a specific observation contract. There are $n \geq 2$ independent, identically distributed complete Gaussian measurement vectors,

$$Y^{(r)} \sim N_p\left(g, \Sigma_Y\right), r = 1, \ldots, n, \tag{9.1}$$

with a common unknown positive-definite covariance. Each vector contains the same sound-window and expansion measurements. Correlation inside a vector may be arbitrary. Different cosmological epochs are coordinates, not independent repeats of the complete experiment. A common cosmic-variance contribution across repeats would require a different model. Gaussianity is assumed for the supplied expansion coordinate; it does not follow automatically by squaring noisy measurements of H. [30; 20]

Let $\overline{Y}$ and S_Y denote the sample mean and covariance. Coordinatewise Student intervals with a Bonferroni allocation give

$$r_j = t_{n-1,1-\alpha/(2p)} \sqrt{\frac{\left(S_Y\right)_{jj}}{n}}, E = \left\{\left|\overline{Y}_j - g_j\right| \leq r_j \text{ for all } j\right\}. \tag{9.2}$$

Each marginal Student pivot is exact under (9.1), and a union bound proves $Pr_g\left(E\right) \geq 1 - \alpha$. Independence between coordinates is unnecessary. This approach sacrifices precision to avoid estimating and inverting the full covariance matrix. The resulting probability statement concerns repeated complete experiments under the declared model; it is not a posterior probability assigned to the fixed physical parameters.

9.2 From window means to a continuous sound band

Let sound centers cover I, with maximum spacing h_s, and let the largest centered uniform window have width w. The centers, widths and gains are calibrated inputs. Suppose

$$\chi \in C^2(I_{ext}), \| \chi'' \|_\infty \le M_\chi, \tag{9.3}$$

on the interval enlarged to contain the complete windows. Interpolate the measured sound means linearly to obtain $\hat{\chi}$. Centered averaging differs from the central value by at most $M_\chi w^2/24$, while linear interpolation between exact node values contributes at most $M_\chi h_s^2/8$. Consequently, on the same event E,

$$\| \chi - \hat{\chi} \|_\infty \le \varepsilon_\chi, \varepsilon_\chi = \max_{j \in sound} r_j + \frac{M_\chi w^2}{24} + \frac{M_\chi h_s^2}{8}. \tag{9.4}$$

These terms are respectively measurement uncertainty, averaging bias and interpolation bias. More repeats shrink the first term. Finer centers and narrower windows can shrink the last two, subject to their actual measurement errors. The bound M_χ defines the model class and is not established merely because the fitted curve is smooth. [20]

The dependence on regularity matters for combining earlier chapters. For a finite harmonic family,

$$\chi(s) = \chi_0 + \sum_k A_k \cos(\omega_k s + \delta_k) \Rightarrow \| \chi'' \|_\infty \le \sum_k |A_k| \omega_k^2. \tag{9.5}$$

At $\omega_g = 2\pi/\log\varphi$, amplitude 0.005 contributes approximately 0.852427 to this bound; amplitude 0.08 contributes approximately 13.638831. The source experiment with $M_\chi = 1$ covers the first example, not the second. Detecting a higher-frequency component therefore does not automatically supply a valid physical confidence set under an earlier, smaller regularity budget. [20; 6]

9.3 Propagating uncertainty without linearizing the sound error

Define $\hat{Q}, \hat{J}$ from $\hat{\chi}$ as in Chapter 5. The exact exponential relation implies

$$e^{-a(s)} \le \frac{Q(s)}{\hat{Q}(s)} \le e^{a(s)}, a(s) = 3\varepsilon_\chi |s). \tag{9.6}$$

Integration in either direction from zero gives

$$|J(s) - \hat{J}(s)) \le |\hat{J}(s))\left[e^{a(s)} - 1\right). \tag{9.7}$$

If a numerical value $\widetilde{J}_i$ has a justified quadrature error ζ_i, use

$$R_i=\left(\left|\widetilde{J}_i\right)+\zeta_i\right)\left[e^{3\varepsilon_\chi|s_i)}-1\right)+\zeta_i. \tag{9.8}$$

The expansion design and its entrywise uncertainty are then

$$\begin{gathered}\widehat{A}_i=\left(1,\widetilde{J}_i,\widetilde{J}_i^2,E_i\right),\\ B_i=\left(0,R_i,2\left|\widetilde{J}_i\right)R_i+R_i^2,0\right),\\ \left|A_\chi-\widehat{A}\right)\le B.\end{gathered} \tag{9.9}$$

The first and last columns are exact because the reference epochs and scales are supplied. The quadratic-column bound follows from $\left|x^2-y^2\right)\le 2\left|y\right)\left|x-y\right)+\left|x-y\right)^2$. No first-order approximation in ε_χ has been made. A quadrature truncation bound and a comparison between two integration grids are different forms of evidence; a fully rigorous numerical enclosure must also account for the numerical errors its guarantee claims to cover. [20]

9.4 A confidence set for the entire fixed-vacuum inverse

For known b and supplied u, let $f_i=h_i^{obs}-b+u\,e^{-2s_i}$ be the measured expansion residual, and let e be its simultaneous error radii from (9.2). The physical coefficient vector is $\theta=\left(\alpha_0,\alpha_1,\alpha_2,d\right)\ge 0$. Define the outer polyhedron

$$P_b=\left\{\theta\ge 0:\left|\widehat{A}\,\theta-f\right)\le e+B\,\theta\right\}. \tag{9.10}$$

Equivalently, its absolute-value condition is the pair of linear inequalities $\left(\widehat{A}-B\right)\theta\le f+e$ and $\left(-\widehat{A}-B\right)\theta\le -f+e$. Nonnegative coefficients make this a linear feasibility problem even though the underlying sound uncertainty is functional. [20]

Let $\Theta_b(g)$ contain every coefficient vector whose one common sound history obeys the regularity and physical assumptions and reproduces the true observation mean g. It may contain one vector or many.

Theorem 9.1 — Same-event coverage. Under the observation and enclosure assumptions,

$$\Pr_g\left\{\Theta_b(g)\subseteq P_b\right\}\ge 1-\alpha. \tag{9.11}$$

Proof. On E, every admissible history sharing the true sound means lies in the band (9.4), hence satisfies (9.9). For any $\theta \in \Theta_b(g)$, choose its witnessing history. Then

$$\begin{aligned} \left|\widehat{A}\,\theta - f\right| &\le \left|g_F - f\right| + \left|\left(\widehat{A} - A_\chi\right)\theta\right| \\ &\le e + B\,\theta. \end{aligned} \tag{9.12}$$

Thus all such vectors are retained together on E. The probability bound follows from (9.2), without assuming rank four or selecting a single true parameter. ▫

The set is deliberately outer. It forgets that the entries of A_χ must arise from the same integrated sound path, so some admitted coefficients may have no common-history witness. Physical filtering and additional valid anchors can tighten it, provided they retain every compatible history. Testing only the nominal fitted sound path would not have that property.

Intersect the set with the positive physical branch and map it through (5.4). This preserves simultaneous coverage of the entire physical identified set. For simple computable bounds, linear programs give coordinate intervals $l_j \le \theta_j \le u_j$. A conservative discriminant enclosure is

$$\Delta_L = max\left(0\,, l_1^2 - 4\,u_0 u_2\right), \Delta_U = u_1^2 - 4\,l_0 l_2. \tag{9.13}$$

Whenever the box contains a physical vector,

$$\begin{gathered} \frac{l_2}{\Delta_U} \le \eta \le \frac{u_2}{\Delta_L}, \\ \frac{S\,\Delta_L}{2\,\kappa_0 u_2} \le \lambda_{bare} \le \frac{S\,\Delta_U}{2\,\kappa_0 l_2}, \\ l_d \le d \le u_d. \end{gathered} \tag{9.14}$$

A zero denominator in an upper bound means no finite upper bound from that formula. The AdS condition and other prior information remain separately identifiable restrictions. Marginal endpoints need not occur at one jointly feasible vector. These intervals condition on the supplied conversion scales; uncertainty in those scales needs its own joint enlargement.

9.5 The correct extension when the vacuum is free

Freeing b changes the intercept domain and the physical map. With measured $y = h + u\,e^{-2s}$, use

$$\begin{aligned} &|\,y_i - \beta_0 - \beta_1 \tilde{J}_i - \beta_2 \tilde{J}_i^2 - d\,E_i\,| \\ &\le e_i + R_i\beta_1 + \left(2\left|\tilde{J}_i\right| R_i + R_i^2\right)\beta_2, \\ &\qquad \beta_0 \in \mathbb{R},\, \beta_1, \beta_2, d \ge 0\,. \end{aligned} \tag{9.15}$$

The physical branch requires $\beta_2 > 0$. The unrestricted sign of β_0 causes no difficulty because its column has no sound error. Imposing $\beta_0 \ge 0$ would exclude some free negative-vacuum models without justification. The proof of simultaneous coefficient retention is otherwise identical. [3]

The physical image now contains every admissible C from (6.3), not one tension obtained from a discriminant. Formally, take the union over retained coefficients and common sound histories in the band, retaining their allowed C intervals. That union includes the full physical identified set on E. A necessary physical enclosure can replace the exact common-history search when computing conservative bounds; a check against only $\hat{J}$ cannot silently replace it.

For coefficient bounds with $0 \le l_1 \le u_1$ and $0 < l_2 \le u_2$, the geometric invariant obeys

$$L_- = \frac{l_1^2}{4\,u_2} - u_0,\; L_+ = \frac{u_1^2}{4\,l_2} - l_0. \tag{9.16}$$

If $L_- > 0$ and $0 \le d_- \le d_+$, then

$$\begin{aligned} \left(S\,L_+\right)^{-1/2} \le \ell \le \left(S\,L_-\right)^{-1/2}, \\ a_*^4\,S\,d_- \le \mu_B \le a_*^4\,S\,d_+, \\ \left(\frac{d_-}{L_+}\right)^{1/4} \le \frac{r_h}{a_*} \le \left(\frac{d_+}{L_-}\right)^{1/4} \quad (k=0). \end{aligned} \tag{9.17}$$

The horizon expression requires a positive-radius physical branch. If L approaches zero at positive d, no finite upper horizon bound follows without an additional restriction, such as the exterior-brane condition. Even sharply determined L, d can coexist with the free C family and an unbounded relative uncertainty in split tension. Total vacuum stress τ_{vac} requires the physical normalization discussed in Chapter 10; it is not obtained by renaming $1/\eta$.

9.6 Rank certification and continuous weak information

Assume at least four expansion epochs. For any positive diagonal 4×4 column scaling W, define

$$\gamma = \sigma_{min}\left(\widehat{A}\, W\right) - \left\| B W \right\|_F . \tag{9.18}$$

On E, $\gamma > 0$ implies every admissible sound history has a full-rank sampled matrix, since $\left\| A_\chi W z \right\| \geq \gamma$ for every unit vector z. This certifies separation from exact Riccati dependence without estimating χ'. A nonpositive margin is inconclusive. A positive margin identifies coefficients at each fixed history; it does not make different uncertain histories share one coefficient vector, and it does not remove the free-vacuum C orbit. [20]

Continuous weak information is a separate observation setting. Suppose simultaneous continuous bands supply $\left| y - \hat{y} \right) \leq \varepsilon_y(s)$ and the sound bound. Choose a test ϕ supported strictly inside every allowed $J(I)$, with $M_3 = \left\| \phi''' \right\|_\infty$ and Lipschitz constant L_3 for ϕ'''. For exact nominal integrals, define

$$\begin{aligned} \widehat{K} &= \phi'''\left(\hat{J}\right)\widehat{Q}, \\ R_J &= \left|\hat{J}\right)\left(e^a - 1\right), \\ R_K &= \widehat{Q}\left\{ e^a L_3 R_J + M_3\left(e^a - 1\right)\right\}, \\ \widehat{A}_\phi &= \int_I \hat{y}\, \widehat{K}\, d\, s\,, \widehat{B}_\phi = \int_I E\, \widehat{K}\, d\, s\,. \end{aligned} \tag{9.19}$$

The kernel difference is bounded by R_K. Therefore, with

$$\begin{aligned} r_A &= \int_I \varepsilon_y\left|\widehat{K}\right) d\, s + \int_I \left(\left|\hat{y}\right) + \varepsilon_y\right) R_K\, d\, s\,, \\ r_B &= \int_I E\, R_K\, d\, s\,, \end{aligned} \tag{9.20}$$

every compatible nonnegative d satisfies

$$\left| \widehat{A}_\phi - d\, \widehat{B}_\phi \right) \leq r_A + d\, r_B . \tag{9.21}$$

This follows by perturbing the exact weak identity $A_\phi = d\, B_\phi$. It remains valid with free vacuum because the constant intercept is annihilated. It is two linear inequalities in d, avoiding division by an uncertain denominator. Valid integration errors can be added to r_A, r_B. Several tests can be intersected on the same continuous-band event without a new multiplicity charge. Four expansion values alone do not provide that event. [19; 20; 3]

9.7 What the computational evidence establishes

Series II XI reports 4,800 synthetic datasets using 24 complete repeats, 65 sound centers and four expansion epochs. Its full outer sets retained the generating coefficients in every reported trial. A conditional comparison that omitted sound uncertainty sometimes retained them much less often, although both procedures' broad marginal boxes covered the generating mass and tension. These results illustrate conservatism and the cost of ignored uncertainty. They establish neither a measured tension nor sharp or uniform empirical coverage; the theorem follows from its observation assumptions and set-inclusion proof. [20]

A projected-sound interval does not supply the uniform sound band needed by the inverse. Combining discovery-error allowance α_T and physical noncoverage allowance α_C guarantees only $1 - \alpha_T - \alpha_C$ by a union bound. Unconditional coverage need not survive selection. Every report therefore needs its observation model, regularity bounds and exact confidence target. [28; 29; 6]

10 Covariant Completion and Vacuum Normalization

10.1 From a reconstructed history to a dynamical question

An inverse reconstruction determines what a chosen class of histories permits us to infer. A dynamical completion asks a further question: which action produces those histories, and which additional predictions follow once its parameters, matter standards, and boundary conditions have been fixed? The distinction matters because a sufficiently adjustable action can reproduce a supplied history without explaining why that history occurs. The final research memorandum makes this distinction explicit and supplies the completion examined in this chapter. [3] [4]

The preceding reconstruction kept the algebraic coefficient $g_{ref} = g_{split} = \lambda / \left(6 M_5^6\right)$ fixed. That coefficient must now be compared with the response to excitations after all constant matter vacuum terms have been removed. The comparison changes the interpretation of a possible gravitational calibration. It leaves the predecessor's algebraic equivalence class intact within its declared normalization convention.

We use $c = \hbar = 1$, the brane signature $(-,+,+,+)$, and a five-dimensional Lorentzian bulk. The induced brane metric is $h_{\mu\nu}$. The brane separates two reflection-related regions M_+ and M_-. A minimal completion is

$$\begin{aligned} S &= S_{bulk} + S_{GHY} + S_{\Sigma}, \\ S_{bulk} &= \frac{M_5^3}{2} \sum_{\sigma=\pm} \int_{M_\sigma} d^5x \sqrt{-g_5} \left(R_5 - 2\Lambda_5\right), \\ S_{\Sigma} &= \int_{\Sigma} d^4x \sqrt{-h} \left[-\lambda + F(X) - U_0 + L_{SM} + L_{DM}\right], \\ X &= -\frac{1}{2} h^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi \, , \Lambda_5 = -\frac{6}{\ell^2}. \end{aligned} \tag{10.1}$$

The boundary action contains the Gibbons–Hawking–York terms on both sides with consistent normal orientations. The convention $8\pi G_5 = M_5^{-3}$ fixes the gravitational normalization. For a scalar assigned its usual mass dimension, $[\phi] = E$, while $[X] = [F] = [U_0] = [\lambda] = E^4$, $[M_5] = E$, and $[\ell] = E^{-1}$. Constant vacuum contributions from the other brane sectors are included in the declared renormalized constant λ; their excitations remain explicit. This choice organizes the terms in the action. It does not explain their renormalized magnitudes. [4]

No induced brane Einstein term, Gauss–Bonnet term, bulk scalar, or direct scalar–Standard Model coupling is present. The resulting junction dynamics therefore belongs to the stated Randall–Sundrum type model. The symmetry content consists of five-dimensional covariance, the Z_2 reflection, the matter gauge symmetries, and the scalar shift $\phi \mapsto \phi + constant$. The shift symmetry permits a function of X; it does not select a particular power or a cosmological charge. [7] [8] [31] [4]

10.2 Field equations and the scalar fluid

Bulk variation and the brane junction condition give

$$G^{(5)}_{AB} + \Lambda_5 g_{AB} = 0 ,$$
$$K^{+}_{\mu\nu} = -\frac{1}{2 M_5^3}\left(T_{\mu\nu} - \frac{T}{3} h_{\mu\nu} \right). \tag{10.2}$$

Here $T_{\mu\nu}$ is the full brane stress, including its constant part. The sign of $K^{+}_{\mu\nu}$ follows the specified normal convention; the reflection condition supplies the factor of one half. These are established junction equations. [7] [8]

For the scalar, integration by parts in its variation and variation with respect to $h^{\mu\nu}$ yield

$$\nabla_\mu \left(F_X \nabla^\mu \phi \right) = 0 ,$$
$$T^{\phi}_{\mu\nu} = F_X \partial_\mu \phi \, \partial_\nu \phi + \left(F - U_0 \right) h_{\mu\nu} . \tag{10.3}$$

On a homogeneous timelike branch, $X = \dot{\phi}^2 / 2 > 0$, this stress has perfect-fluid form. Its density, pressure, and local scalar kinetic coefficients are

$$p_\phi = F - U_0 , \rho_\phi = 2 X F_X - F + U_0 ,$$
$$K = F_X + 2 X F_{XX} , c_\phi^2 = \frac{F_X}{K} . \tag{10.4}$$

The homogeneous equation is $K \ddot{\phi} + 3 H F_X \dot{\phi} = 0$, and the conserved shift charge is $a^3 F_X \dot{\phi}$. Positivity of F_X and K gives the local spatial and temporal kinetic signs on this background. The identification with an irrotational barotropic fluid and the distinction between an adiabatic sound speed and the propagation speed of a general scalar are established results. For a purely kinetic action the relevant speeds agree on the branch under consideration. [32] [33]

10.3 A constructive local realization theorem

The availability of a scalar action is a weak selection criterion for the histories used in the inverse problem. The following construction demonstrates why. Let

$s=\ln\left(a/a_{*}\right)$, with $s=0$ chosen inside the interval of interest. Suppose a differentiable barotropic history satisfies positive density, positive enthalpy $W=\rho+p$, conservation $\rho'=-3W$, and a continuous sound history $0<\chi=d\,p/d\,\rho\le 1$ on a compact interval.

Proposition 10.1. For every $X_{*}>0$, the assignments

$$\begin{aligned} X(s)&=X_{*}\exp\left[-6\int_{0}^{s}\chi(t)\,dt\right],\\ P\big(X(s)\big)&=p(s) \end{aligned} \tag{10.5}$$

define a local purely kinetic action reproducing that density and sound history, with positive local kinetic coefficients. [4]

Proof. Since $X'=-6\chi X<0$, the map from s to X is invertible on the interval. Conservation gives $p'=-3\chi W$, so differentiation along the trajectory gives

$$\begin{aligned} P_{X}&=\frac{p'}{X'}=\frac{W}{2X}>0,\\ 2X\,P_{X}-P&=\rho,\\ P_{X}+2X\,P_{XX}&=\frac{P_{X}}{\chi}>0. \end{aligned} \tag{10.6}$$

For the last identity, differentiate the reconstructed density with respect to X and use $d\,p/d\,\rho=\chi$. Finally, $W'=-3(1+\chi)W$, and the logarithmic derivative of $a^{3}W/\sqrt{2X}$ vanishes. This quantity equals $a^{3}P_{X}\sqrt{2X}$, so the homogeneous scalar equation holds as well.

The construction fixes the action only on the visited interval of X. It supplies neither an extension to a zero-gradient vacuum nor a principle selecting the observed history. The freedom in X_{*} also exposes field normalization: choosing a coordinate on scalar-field space is not an absolute density measurement. The broader relation between kinetic fields and barotropic fluids predates this construction; the point here is its consequence for the inverse research program. [33] [34] [35]

10.4 Constant vacuum splitting is an exact redundancy

Separate the total constant vacuum stress from its excitations:

$$\begin{aligned} \tau&=\lambda+U_{0}>0,\\ \rho_{e}&=2X\,F_{X}-F+\rho_{SM}+\rho_{DM},\\ p_{e}&=F+p_{SM}+p_{DM}. \end{aligned} \tag{10.7}$$

The scalar τ denotes a vacuum stress, not the tensor often denoted by that letter in brane references. If the action extends to a zero-gradient vacuum, we adopt $F(0)=0$ by absorbing its constant into U_0. A local action supplied by Proposition 10.1 need not admit that extension.

Proposition 10.2. For any allowed constant Δ, the transformation

$$\lambda \mapsto \lambda + \Delta, U_0 \mapsto U_0 - \Delta \tag{10.8}$$

leaves the full action unchanged.

Proof. Its only dependence on these constants is through $-\lambda - U_0 = -\tau$. Every field variation is therefore identical before and after the relabeling.

No observable of this identical action separates the two constants. A microscopic model might distinguish their origins through additional structure, but the present parameterization contains no such information. This exact statement makes concrete the vacuum–matter split ambiguity noted in the foundational brane analysis. It is a redundancy of description, not a newly discovered physical symmetry. [7] [4]

10.5 Overall action scaling is a different freedom

For the predecessor's one-fluid realization, let $\alpha > 0$ label the normalization orbit:

$$\begin{aligned} \lambda_\alpha &= \alpha^2 \lambda, M_{5,\alpha}^3 = \alpha M_5^3, \\ F_\alpha &= \alpha F, U_{0,\alpha} = \alpha U_0 - (\alpha^2 - \alpha)\lambda. \end{aligned} \tag{10.9}$$

It follows that $\tau_\alpha = \alpha \tau$ and $\rho_{e,\alpha} = \alpha \rho_e$. With geometric Λ_5 fixed, all bulk, boundary, and one-sector brane terms scale together, giving $S_\alpha = \alpha S$. The classical stationary configurations are unchanged because multiplying an action by a nonzero constant leaves its Euler–Lagrange equations unchanged. [3] [4]

This transformation differs from Equation 10.8: the action is multiplied, rather than left identical. At fixed $\hbar$, its quantum phase changes. Furthermore, an orbit that scales all brane matter does not remain available unchanged when independently normalized Standard Model or other matter sectors are held fixed. A physical energy standard can consequently obstruct this particular scaling. That observation does not establish identifiability across every enlarged dark-sector action.

10.6 Background coupling and local gravitational matching

The homogeneous junction equation, written in the vacuum-subtracted variables, is

$$H^2+\frac{k}{a^2}=\frac{\left(\tau+\rho_e\right)^2}{36\,M_5^6}-\frac{1}{\ell^2}+\frac{\mu_B}{a^4},\mu_B\geq 0. \tag{10.10}$$

Expanding the square defines a background response coefficient and an effective vacuum curvature:

$$\begin{aligned} g_{bg}&=\frac{\tau}{6\,M_5^6}=M_4^{-2},\\ \frac{\Lambda_{eff}}{3}&=\frac{\tau^2}{36\,M_5^6}-\ell^{-2}. \end{aligned} \tag{10.11}$$

M_4 here is defined by a background coefficient. Identifying g_{bg} with locally measured $8\pi G_N$ additionally requires a localized gravitational response, calibrated matter standards, and negligible extra scalar or bulk contributions in the measurement regime. The tuned weakly curved vacuum supplies the familiar localization limit; a rolling scalar cosmology requires its own matching calculation. The present source has not solved that localized-source problem. [31] [4]

The two coefficients transform differently:

$$\begin{aligned} g_{ref}&=g_{split}=\frac{\lambda}{6\,M_5^6},\\ \left(g_{split}\right)_\alpha&=g_{split},\left(g_{bg}\right)_\alpha=\frac{g_{bg}}{\alpha}. \end{aligned} \tag{10.12}$$

Thus an exact nonzero determination of g_{bg} in fixed absolute stress units selects at most one member of this orbit, provided the matching just described is justified. It cannot distinguish λ from U_0 under the identical-action transformation. The conserved split coefficient used in Chapters 2–9 and the operationally matched coefficient answer different questions once a separate constant matter vacuum term is present.

If ℓ, Λ_{eff}, and a matched M_4 are independently supplied, algebra gives

$$\begin{aligned} \tau&=6\,M_4^2\left(\ell^{-2}+\frac{\Lambda_{eff}}{3}\right),\\ M_5^6&=\frac{\tau\,M_4^2}{6}. \end{aligned} \tag{10.13}$$

The bracket must be positive. These three inputs have not been jointly measured for the model studied here; an effective vacuum term from a different cosmological fit cannot be imported without its assumptions. The result identifies the extra information needed to determine the total vacuum stress, while leaving the separation of its constant pieces redundant. It supplies a precise interface between the formal inverse result and a possible experiment. [4]

10.7 Which additional assumptions carry information

The different normalization choices can now be compared without treating every supplied constant as a new measurement.

Table 10.1. Information needed to fix an action normalization.

Supplied information	What it fixes	What remains
A sound history $\chi(s)$	The allowed fluid shape	Density normalization and vacuum offset
Existence of a kinetic $P(X)$	A local field realization	Choice of action outside the sampled interval
A named cubic coefficient	A field-coordinate convention unless other couplings are fixed	Charge and absolute stress calibration
A matched nonzero g_{bg}	At most one representative of the stated scaling orbit	Other action families and the constant split
Matched M_4, ℓ, and Λ_{eff}	Total τ and M_5	Separate values of λ and U_0

This separation also disciplines proposed additional principles. None of the symmetries listed in Section 10.1 selects a golden-ratio frequency, a preferred kinetic power, or the Noether charge. A proposed numerical relation would have to follow from additional dynamics or independently specified boundary data. Reproducing a chosen relation by adjusting an otherwise unrestricted function leaves the selection problem unanswered. The completion establishes what is dynamically possible within the class; a predictive restriction must further explain which members are admitted before the target history is imposed. [4]

11 Kinetic Actions and Local Consistency

11.1 Constant sound speed and an affine equation of state

A kinetic realization should first reproduce the intended fluid and then be examined on backgrounds different from the one used to construct it. The constant-sound example makes both steps explicit. Choose $F(X)=A\,X^{n}$ with $A>0$, $n\geq 1$, and a homogeneous timelike field. The coefficient has dimension $[A)=E^{4-4n}$. Substitution into the density and pressure gives

$$\begin{aligned}\rho_{\phi}&=(2n-1)\,A\,X^{n}+U_{0},\\ p_{\phi}&=A\,X^{n}-U_{0}=\frac{\rho_{\phi}-2nU_{0}}{2n-1},\\ c_{\phi}^{2}&=\frac{1}{2n-1}.\end{aligned}\tag{11.1}$$

The excitation density $\rho_{\phi,e}=(2n-1)\,A\,X^{n}$ has pressure $p_{\phi,e}=A\,X^{n}$. Conservation therefore makes it redshift as $a^{-3(1+c_{\phi}^{2})}$. The constant U_{0} supplies the affine intercept; it does not alter the kinetic sound speed. This established field–fluid relation realizes the predecessor's $\chi=0.2$ example with $n=3$ and $A=M^{-8}$. [34] [4]

The conserved current also determines the background normalization. For a chosen sign of $\dot{\phi}$, $a^{3}nA\,X^{n-1}\sqrt{2X}$ is constant. Consequently $X\propto a^{-6/(2n-1)}$, and raising it to the power n reproduces the density law. The current's value is independent initial information. Giving the coefficient the name M^{-8} does not fix that information or attach an external energy calibration to the field.

The radiation-like case illustrates a connection with the rank exceptions of the inverse reconstruction. At $\chi=1/3$, one has $n=2$, and the excitation density scales as a^{-4}. Its linear background dependence can overlap the dark-radiation column of the reconstruction. Rewriting the fluid as a scalar does not create a new independent column or remove this structural degeneracy. [3] [4]

11.2 What the kinetic signs test

Consider a small fluctuation π about a local slowly varying background $\dot{\phi}$, so that $\phi = \dot{\phi} + \pi$. In a local inertial patch with homogeneous background gradient $v = \dot{\dot{\phi}}$ and $X_0 = v^2/2$, the kinetic invariant changes by

$$X = X_0 + v\,\dot{\pi} + \frac{1}{2}\left[\dot{\pi}^2 - (\nabla \pi)^2\right]. \tag{11.2}$$

At the order relevant to the principal kinetic terms, background derivatives and metric mixing are held aside. Expanding a general F gives

$$L_2 = \frac{1}{2}\left[\left(F_X + 2\,X_0 F_{XX}\right)\dot{\pi}^2 - F_X(\nabla \pi)^2\right)_{X_0}. \tag{11.3}$$

The local Fourier equation consequently has $\omega^2 = \left(F_X / K\right)|k|^2$. A positive temporal coefficient avoids a negative kinetic energy for this scalar fluctuation; a positive spatial coefficient avoids the corresponding short-wavelength gradient instability. Requiring their ratio not to exceed one imposes the stated subluminal condition relative to the local brane metric in this timelike setting. These are tests of a particular local principal part, rather than a proof about every state of the coupled five-dimensional system. [32] [4]

This distinction is useful even for a spatially homogeneous cosmology. Positive coefficients do not tell us whether nonlinear waves form caustics, whether all admissible initial data evolve globally, or whether the scalar couples to an unstable bulk mode. Those questions require equations and boundary conditions beyond the frozen-background diagnostic. Studies of nonlinear kinetic theories make the need for additional control clear, although their different actions and applications cannot be imported as solutions of the present model. [36]

11.3 The cubic example has a degenerate zero-gradient vacuum

For $F = X^3/M^8$, $F_X = 3\,X^2/M^8$ and $K = 15\,X^2/M^8$. Both are positive for every $X > 0$, while both vanish at zero. The behavior at that endpoint can be seen without relying on the ratio of two vanishing coefficients.

Write $\delta X_1 = v\,\dot{\pi}$ and $\delta X_2 = \left[\dot{\pi}^2 - \left(\nabla \pi\right)^2\right]/2$. In the expansion of $\left(X_0 + \delta X_1 + \delta X_2\right)^3$, the quadratic terms are $3 X_0^2 \delta X_2 + 3 X_0 \delta X_1^2$. The cubic terms are $6 X_0 \delta X_1 \delta X_2 + \delta X_1^3$. Using $v = \sqrt{2 X_0}$ yields

$$L_2 = \frac{X_0^2}{2 M^8}\left[15\,\dot{\pi}^2 - 3\left(\nabla \pi\right)^2\right],$$
$$L_3 = \frac{X_0 \sqrt{2 X_0}}{M^8}\left[5\,\dot{\pi}^3 - 3\,\dot{\pi}\left(\nabla \pi\right)^2\right]. \tag{11.4}$$

At $X_0 = 0$, the action begins at sixth order in derivatives of the fluctuation amplitude, since X itself is quadratic in $\partial \pi$. There is no quadratic operator to invert into a Gaussian scalar propagator in the pure cubic truncation. A finite limiting value of $F_X / K = 1/5$ does not repair the missing kinetic normalization. [4]

For $X_0 > 0$, canonical normalization of the temporal term gives

$$\pi_c = \frac{\sqrt{15\,X_0}}{M^4}\,\pi\,,$$
$$L_3 \sim \frac{M^4}{X_0^{3/2}}\left(\partial \pi_c\right)^3, \Lambda_{int} \sim \frac{X_0^{3/4}}{M^2}. \tag{11.5}$$

The last relation is dimensional power counting for the displayed interaction, with numerical and angular coefficients suppressed. It is not a completed unitarity calculation, particularly once gravity and other operators are included. It shows why no background-independent weak-coupling scale extending to $X_0 = 0$ has been established. [4]

The implication for a finite-density universe is narrower. A background-dependent scale can remain above the frequencies of a particular cosmological calculation even as it tends to zero in a different limit. The correct comparison is between the frequencies and gradients actually used, the background-dependent interactions, and the operators retained in the action. The local-vacuum defect therefore defeats an unqualified claim of a regular perturbative vacuum theory, without by itself disproving every finite-density solution.

11.4 Adding a canonical term changes the fluid

A simple comparison restores a nonzero quadratic operator at zero gradient:

$$P\left(X\right) = X + \frac{X^3}{M^8} - U_0\,, x = \frac{X}{M^4}. \tag{11.6}$$

The excitation density, pressure, and kinetic coefficients become

$$\rho_{\phi,e}=M^4\left(x+5x^3\right),\ p_{\phi,e}=M^4\left(x+x^3\right),$$
$$P_X=1+3x^2,\ K=1+15x^2,$$
$$c_\phi^2=\frac{1+3x^2}{1+15x^2},\ \frac{1}{5}<c_\phi^2\le 1. \tag{11.7}$$

Here $x\ge 0$ describes the homogeneous timelike branch and its zero-gradient limit. At zero, the scalar fluctuation has a canonical kinetic term and luminal local propagation. The fluid four-velocity, which divides by $\sqrt{2X}$, need not be used at that endpoint. At large positive x, the squared sound speed approaches $1/5$, but it is no longer the exact constant history of the cubic model. [4]

The equation-of-state ratio of the excitations also changes: $w_e=\left(1+x^2\right)/\left(1+5x^2\right)$. At small x, both the pressure–density ratio and the propagation speed approach the canonical stiff-fluid limit. The system therefore interpolates between two kinetic regimes rather than preserving a single affine history. Repairing a local property of an action can alter the inverse data it was designed to reproduce.

11.5 Evolution from the equation of motion and from the current

Since $\dot{X}=\dot{\phi}\ddot{\phi}$, the homogeneous scalar equation implies $dX/ds=-6XP_X/K$. For the canonical-plus-cubic action this gives two equivalent descriptions:

$$\frac{dx}{ds}=-6x\frac{1+3x^2}{1+15x^2},$$
$$a^3\left(1+3x^2\right)\sqrt{2x}=constant. \tag{11.8}$$

An overall factor M^2 has been absorbed into the charge constant. On the positive branch, squaring the current relation between $s=0$ and s gives the useful algebraic form

$$x(s)\left[1+3x(s)^2\right]^2=x(0)\left[1+3x(0)^2\right]^2e^{-6s}. \tag{11.9}$$

The left-hand side increases strictly for $x>0$, so this equation selects a unique positive root. It offers a numerical comparison independent of integrating the differential equation for $\ln x$. In the small-x regime, x approximately redshifts as a^{-6}; in the cubic-dominated regime it approximately redshifts as $a^{-6/5}$. Their corresponding excitation densities approach a^{-6} and $a^{-18/5}$.

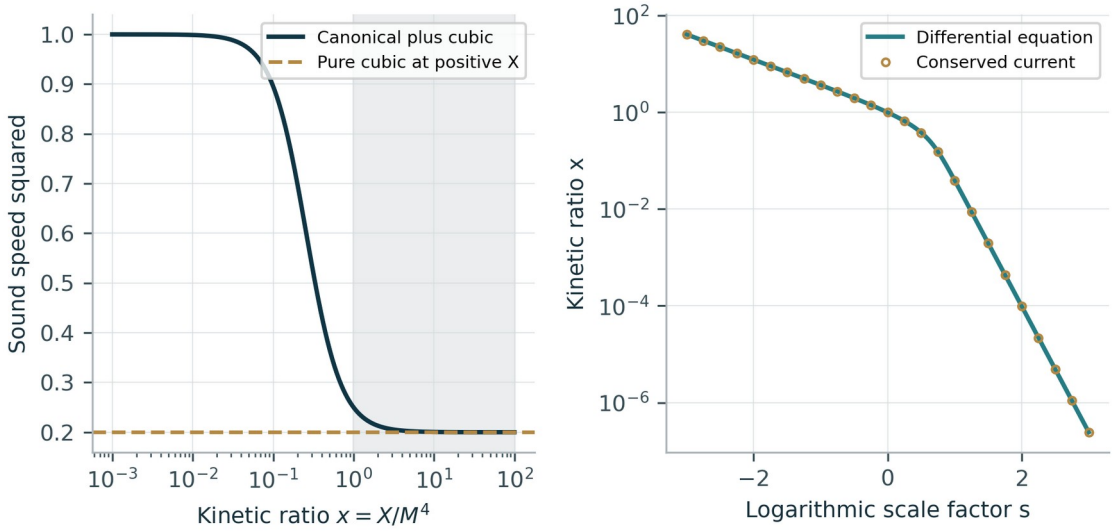

Figure 11.1. Left: the canonical-plus-cubic sound speed and the positive-X pure-cubic limit. The shaded x>1 range needs separate effective-theory justification. Right: homogeneous evolution from x(0)=1, computed by differential integration and an independent conserved-current root. These are model calculations, not fitted observations.

The figure compares the two sound-speed functions and a current-conserving trajectory initialized with $x(0)=1$. The calculation made for this edition compares differential evolution with an independent current-root solution on $-3 \le s \le 3$. Their maximum difference in $\ln x$ is approximately 1.48×10^{-11}, with relative current drift 1.91×10^{-11}; Appendix D records the program and its scope. The final source paper reports its own comparison over $-3 \le s \le 3$, with maximum difference 6.91×10^{-10} and relative drift 3.90×10^{-10}. Those separate source-reported diagnostics are retained as such. [4]

11.6 What canonical normalization does and does not fix

Setting $P_X(0)=1$ fixes a useful field-coordinate convention around the vacuum. It does not by itself determine the physical value of M, the cosmological charge, or the gravitational normalization. The action remains a model family until those parameters are attached to declared measurements or theoretical restrictions.

The normalization freedom has a concrete expression. Multiply the scalar action by the positive orbit factor α and introduce $\psi = \sqrt{\alpha}\,\phi$, so that $X_\psi = \alpha X$. Its Lagrangian can again have a unit canonical coefficient:

$$\alpha\left(X + \frac{X^3}{M^8} - U_0\right) = X_\psi + \frac{X_\psi^3}{\alpha^2 M^8} - \alpha U_0 . \tag{11.10}$$

The higher-order scale has changed to $\alpha^{1/4}M$. Thus canonical field normalization alone does not prohibit the overall classical rescaling when the other parameters are free to transform with it. Fixing a higher-order coefficient through an independent physical process, or fixing matter interactions that do not admit this rescaling, supplies information of a different kind. This algebra illustrates the distinction in Chapter 10 rather than claiming that every possible dark-sector degeneracy has the same form.

Shift symmetry permits X^2/M^4 and higher powers as well as the terms retained here. If M is interpreted as a generic expansion scale, the large-x region needed to approximate the pure cubic response is not controlled merely by truncating that series. A theory with a justified resummation or a different background cutoff might support such a regime; neither has been derived in the sources used here. [4]

The completed local result is therefore specific. A pure cubic action realizes the constant-sound background and has positive coefficients on its positive-X branch. Its zero-gradient vacuum lacks a quadratic propagator. A canonical addition restores that operator while changing the sound history. These conclusions identify which further dynamics must be supplied before the same model can be used for a full perturbation calculation or an observational likelihood.

12 Physical Scales, Acceleration, and Varying Tension

12.1 The size of the direct quadratic term

A background correction can be mathematically identifiable and still be far below an observationally useful scale. The junction equation provides a direct comparison once the total vacuum stress and the matter normalization are fixed. Expanding Equation 10.10 gives

$$H^2+\frac{k}{a^2}=\frac{\Lambda_{eff}}{3}+\frac{\rho_e}{3M_4^2}\left(1+\frac{\rho_e}{2\tau}\right)+\frac{\mu_B}{a^4}. \tag{12.1}$$

Define the excitation contribution $H_\rho^2=\rho_e/\left(3M_4^2\right)$. The ratio of the direct quadratic density term to the corresponding linear term is

$$\begin{aligned}\epsilon_{brane}&=\frac{\rho_e}{2\tau}\\&=\frac{H_\rho^2\ell^2}{4\left(1+\Lambda_{eff}\ell^2/3\right)}.\end{aligned} \tag{12.2}$$

The second line follows by substituting Equation 10.13; it introduces no additional gravitational law. In SI units, the numerator contains $\left(H_\rho\ell/c\right)^2$. The denominator is positive in the declared $\tau>0$ domain. This relation connects the scale of the correction with the bulk curvature and the vacuum-subtracted gravitational response. [4]

The denominator contains $\tau=\lambda+U_0$. Using a freely split bare λ here would attach an apparent effect size to a quantity that changes under an identical-action relabeling. The bare tension can replace τ only after adopting a convention in which all constant brane vacuum terms have been absorbed into it.

For a precise comparison of expansion rates, let H_{lin} denote the rate in the same flat background with the quadratic term omitted and all other contributions held fixed. If the effective vacuum and dark-radiation contributions are nonnegative, define $f_\rho=H_\rho^2/H_{lin}^2\le 1$. Then

$$\frac{H-H_{lin}}{H_{lin}}=\sqrt{1+f_{\rho}\epsilon_{brane}}-1 \\ =\frac{f_{\rho}\epsilon_{brane}}{\sqrt{1+f_{\rho}\epsilon_{brane}}+1}\leq\frac{\epsilon_{brane}}{2}. \tag{12.3}$$

The rationalized expression also avoids subtracting nearly equal numbers when the correction is tiny. A background with negative compensating contributions does not satisfy this particular bound. Neither does the calculation bound every scalar, dark-radiation, or bulk perturbation effect.

12.2 A declared present-density benchmark

Use the reference inputs $H_0=70\,km\,s^{-1}\,Mpc^{-1}$ and $M_4=2.435\times10^{18}\,GeV$, with the matching assumptions of Chapter 10. They give $H_0=2.26855\times10^{-18}\,s^{-1}$ and the conventional density scale $3M_4^2H_0^2=3.96594\times10^{-47}\,GeV^4$. Setting the excitation density equal to that scale and choosing $\tau^{1/4}=1\,TeV$ gives

$$\epsilon_{brane}=1.983\times10^{-59},\frac{H-H_{lin}}{H_{lin}}\leq9.915\times10^{-60}. \tag{12.4}$$

These numbers are evaluations of declared inputs, not estimates of a measured scalar density. The choice of critical-density scale does not simultaneously fit all other terms in the cosmological equation. With independently fixed nonnegative contributions it is a useful upper reference scale; arbitrarily large excitation densities hidden by negative cancellations require a different analysis. [4]

The TeV value denotes the fourth root of the total vacuum stress. It is neither the five-dimensional Planck mass M_5 nor the kinetic interaction scale M introduced in Chapter 11; those quantities enter different operators and have different matching relations.

The vacuum-tuned conversion $\Lambda_{eff}=0$ gives $\ell=\sqrt{6}\,M_4/\sqrt{\tau}$. A TeV tension scale corresponds to $\ell\simeq1.177\,mm$, while the illustrative choice $\ell=0.1\,mm$ corresponds to $\tau^{1/4}\simeq3.431\,TeV$. These are conversions between parameters. Compatibility with a laboratory experiment requires the force profile predicted by the specified model, convolved with the experiment’s source and detector geometry.

For example, Lee and collaborators tested separations from $52\,\mu m$ to $3.0\,mm$ and reported a limit for gravitational-strength Yukawa interactions. Their smallest separation is not itself a bound on an AdS radius; a Yukawa profile and the Randall–Sundrum correction are different response functions. The experimental result is evidence about its stated force model and statistical analysis. [37]

The reverse calculation is informative. Requiring $\epsilon_{brane} = 0.01$ at the same density gives $\tau^{1/4} \simeq 6.673\, meV$. Under the zero-effective-vacuum conversion, this corresponds to $\ell \simeq 2.643 \times 10^{25}\, m$, or about $0.86\, G\, pc$. Such a scale is incompatible with treating laboratory and ordinary astrophysical gravity as the conventional short-distance localization limit of this simple model. The calculation isolates the difficulty with this direct late-time quadratic route; modifications of the bulk or brane action change the question. [4]

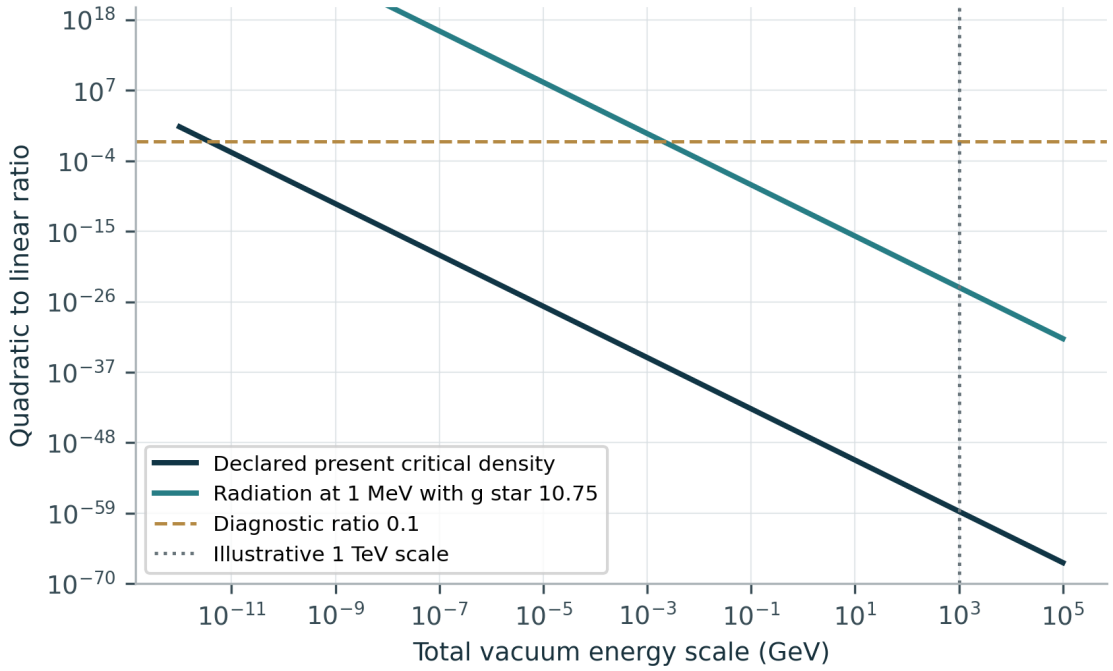

Figure 12.1. The direct quadratic-to-linear ratio as the total vacuum energy scale varies, at two fixed reference densities. The present-density and MeV-radiation curves use the explicit Appendix D inputs. The horizontal ratios are diagnostic levels and the vertical tension mark is a benchmark; none is a fitted exclusion boundary.

The hierarchy plot shows the direct ratio at two fixed reference densities as the total vacuum energy scale varies. Its curves are generated consequences of Equation 12.2. The benchmark and diagnostic tolerance lines identify chosen inputs, not fitted exclusion boundaries.

12.3 The early high-density window

Radiation at temperature T has the idealized density $\rho_r = \pi^2 g_* T^4 / 30$. For a fixed effective relativistic degree count g_*, imposing a diagnostic maximum ratio or solving for the temperature of a chosen ratio gives

$$\begin{aligned} \rho_r(T) &= \frac{\pi^2}{30} g_* T^4, \\ \tau &\geq \frac{\pi^2 g_* T^4}{60\,\epsilon_{max}}, \\ T(\epsilon) &= \left(\frac{60\,\epsilon\,\tau}{\pi^2 g_*} \right)^{1/4}. \end{aligned} \tag{12.5}$$

At $T = 1\,MeV$, $g_* = 10.75$, and $\epsilon_{max} = 0.1$, the resulting tension energy scale is $2.051\,MeV$. It is a radiation-density exercise with an assumed tolerance, rather than a nucleosynthesis likelihood. The MeV scale associated with this requirement must be distinguished from the much stronger scales motivated by short-distance gravity in a conventional RSII interpretation. [38] [4]

Table 12.1. Physical scales under the stated benchmark inputs.

Declared calculation	Result
Present reference density $3 M_4^2 H_0^2$	$3.966 \times 10^{-47}\,GeV^4$
Present direct ratio at $\tau^{1/4} = 1\,TeV$	1.983×10^{-59}
Ratio at $T = 1\,MeV$ and TeV tension	1.768×10^{-24}
Tension scale for the stated BBN tolerance	$2.051\,MeV$
Temperature for $\epsilon = 0.1$, $g_* = 106.75$	$274.7\,GeV$
Temperature for $\epsilon = 1$, $g_* = 106.75$	$488.5\,GeV$
Tuned AdS radius at TeV tension	$1.177\,mm$

The several-hundred-GeV values locate a regime where a high-density correction can matter. They do not determine a primordial spectrum. In such a regime, thermal production of bulk gravitons can carry energy away from the brane, so the

vacuum-bulk, no-exchange model may cease to supply the correct background equations. This energy-transfer issue has been studied in the early brane-cosmology literature and must be incorporated in any proposed application of the window. [38]

12.4 An acceleration obstruction

For constant τ and no bulk exchange, the total excitation stress obeys $\dot{\rho}_e + 3H(\rho_e + p_e) = 0$. Differentiate Equation 12.1 on an interval where $H \neq 0$ and use this conservation equation. The result is

$$\dot{H} - \frac{k}{a^2} = -\frac{\rho_e + p_e}{2M_4^2}\left(1 + \frac{\rho_e}{\tau}\right) - \frac{2\mu_B}{a^4}. \tag{12.6}$$

Adding H^2 eliminates spatial curvature and gives

$$\frac{\ddot{a}}{a} = \frac{\Lambda_{eff}}{3} - \frac{\rho_e + 3p_e + (\rho_e/\tau)(2\rho_e + 3p_e)}{6M_4^2} - \frac{\mu_B}{a^4}. \tag{12.7}$$

Proposition 12.1. If $\tau > 0$, $\rho_e \geq 0$, $p_e \geq 0$, $\mu_B \geq 0$, and $\Lambda_{eff} \leq 0$, then $\ddot{a} \leq 0$. The inequality is strict whenever a negative contribution is nonzero. [4]

Proof. Each term on the right-hand side of Equation 12.7 is nonpositive under the premises. The curvature term has cancelled, so choosing a different spatial curvature does not reverse the conclusion. At an isolated point with $H = 0$, the expression applies by continuity where the complete field equations admit that extension.

Both kinetic examples in Chapter 11 have nonnegative excitation pressure. Ordinary radiation and pressureless baryonic or dark matter preserve the premise. Their positive quadratic correction therefore cannot supply acceleration without a positive effective vacuum contribution or a change in some other premise. A constant U_0 can affect Λ_{eff}, but this is the already declared vacuum sector, rather than acceleration generated by positive-pressure kinetic energy.

Negative pressure in another sector, negative dark radiation, induced gravity, a bulk field, or energy exchange would change the assumptions and require new equations. The proposition is an obstruction for the model actually derived, not a classification of all higher-dimensional cosmologies.

12.5 A varying tension must enter a conserved action

Replacing a constant by a chosen function of time does not complete a covariant mechanism. Within a restricted scalar-dependent brane action, write

$$\begin{aligned} L_{\Sigma} &= F(X) - U(\phi) - \lambda(\phi) + L_{other}, \\ V_{eff}(\phi) &= U(\phi) + \lambda(\phi). \end{aligned} \tag{12.8}$$

The homogeneous scalar equation depends on the sum: $K\ddot{\phi} + 3HF_X\dot{\phi} + V_{eff,\phi} = 0$. Calling one part of this potential a varying tension creates no extra dynamical degree of freedom when the action supplies no additional coupling that distinguishes the functions. It also changes the scalar shift symmetry unless the sum is constant. [4]

Suppose the quantity called matter includes $F - U + L_{other}$ but excludes λ. Its pressure and density then satisfy

$$\dot{\rho}_m + 3H(\rho_m + p_m) = -\dot{\lambda} \tag{12.9}$$

in the vacuum bulk without exchange. The total density includes λ, its pressure includes $-\lambda$, and total conservation directly produces the right-hand side. Imposing separate conservation on the same ρ_m while varying λ would remove the energy needed for this identity. A covariant model can include the appropriate exchange; it must declare it. Variable-tension brane models have established antecedents and do not inherit the constant-tension inverse unchanged. [39]

The term $-\dot{\lambda}$ in this definition is an exchange between the parts of the brane stress called matter and vacuum. It does not require a flux into the bulk: that flux was excluded when deriving the equation. If the functions U and λ are relabeled while keeping their sum fixed, the total stress remains the same even though the bookkeeping of this internal exchange changes. A separately modeled bulk flux would add further terms and physical information. This distinction prevents a change of parameter names from being mistaken for a mechanism of energy loss.

Likewise, a fixed M_5 and a varying split λ imply a varying algebraic g_{split}. Calling it an instantaneous Newton constant does not solve the response of a localized source in the rolling background. The resulting research choice is concrete: determine that local response with normalized matter, or formulate an early high-density model with its energy transfer and perturbations. Neither route follows from an arbitrary time dependence inserted into the background equation.

13 Observational Interfaces

13.1 An observable needs a forward model

The inverse results specify information that would identify particular mathematical quantities. An experiment supplies a different starting point: photon arrival times, frequency ratios, angular positions, forces, or statistical correlations in a catalogue. A physical conclusion requires a forward map connecting the action and its boundary data to those recorded quantities. Calibration, source structure, nuisance parameters, and covariance belong to that map. [3] [40] [4]

This distinction separates three questions. Structural identifiability asks whether ideal data determine a parameter within a declared model class. Statistical inference asks how well finite, noisy data determine it. Physical interpretation asks whether the selected parameterization corresponds to a distinguishable interaction or merely another representation of the same predictions. The finite-reference scalar example in Appendix C and the brane equivalence family answer different versions of these questions.

For the brane reconstruction, a supplied continuous sound history is part of the mathematical input. A distance survey does not automatically supply that function. For the kinetic completion, an action specifies a relation between pressure, density, and scalar perturbations, but its normalizations and initial charge must still be fixed. These two levels become an observational theory only after the source and measurement models are joined.

13.2 The local gravitational matching problem

The definition $g_{bg} = M_4^{-2}$ in Chapter 10 is the coefficient of the vacuum-subtracted linear density term. Even the homogeneous response away from the vacuum contains another density-dependent factor:

$$\left.\frac{\partial H^2}{\partial \rho_e}\right)_{a,\tau} = \frac{1}{3M_4^2}\left(1+\frac{\rho_e}{\tau}\right), \tag{13.1}$$

with the other background terms held fixed. This derivative is not a Poisson equation for an isolated mass. It illustrates why reading a coefficient from the homogeneous Friedmann equation does not, by itself, complete a Cavendish or orbital calibration. [4]

In the standard single positive-tension brane vacuum, localization recovers four-dimensional gravity at leading order. The explicit weak-field analysis of Garriga

and Tanaka treats isolated matter and the bulk gravitational response, including its leading Kaluza–Klein corrections. That calculation supplies an appropriate antecedent for local matching; it does not automatically solve the same problem on an arbitrary rolling $P(X)$ background. [41]

A matching calculation for the present completion would specify the background scalar and vacuum state, introduce a normalized localized source, solve the coupled linearized bulk and brane equations with physical boundary conditions, and extract the force and clock response in an overlap region. The source mass must be expressed in the same matter standards used by the experiment. Only then can a parameter inferred from those responses be compared with g_{bg}.

If the required matching reduces the local coefficient to g_{bg}, an absolute calibration can select a representative of the scaling orbit. If additional scalar or bulk contributions remain, the response involves further parameters. In either case, no local measurement distinguishes the identical-action split $\lambda + \Delta$, $U_0 - \Delta$. The distinction between these two normalization questions survives every improvement in instrumental precision. [4]

13.3 Sound speed does not close the bulk perturbations

For a purely kinetic scalar on its timelike branch, the pressure and density functions determine its intrinsic scalar sound speed. They do not determine all gravitational degrees of freedom outside the brane. The projected brane equations contain the bulk Weyl contribution, whose evolution generally requires bulk information. An initially homogeneous dark-radiation coefficient does not specify all of its inhomogeneous perturbations. [7]

The limitation should be stated precisely. Some large-scale curvature evolution can be obtained from brane conservation equations without solving the entire bulk. Langlois, Maartens, Sasaki, and Wands show how this works and also identify the remaining difficulty: relating the curvature perturbations to metric potentials requires Weyl anisotropic stress, which is not determined by the brane equations alone. That missing information matters for cosmic microwave background predictions. [14]

A prospective cosmological calculation must therefore choose the matter content and initial perturbations, specify the allowed bulk modes and boundary conditions, and evolve the observables appropriate to the data set. Multiple separately normalized fluids may also introduce relative perturbations beyond the one-fluid

reconstruction. The local fluid–scalar equivalence of Chapter 10 is valuable within this calculation, but it does not supply its missing gravitational boundary problem.

13.4 What the clock and ranging source actually establishes

The longer scalar–tensor manuscript develops a separate four-dimensional model with matter coupled to a conformally related physical metric. Its clocks, lenses, and massive tracers probe that metric and its source-dependent scalar field. The brane scalar in Equation 10.1 has a different action and coupling structure. The shared subject is inference from calibrated observables; no common dynamical embedding has been derived. [40] [4]

The scalar–tensor source contains a finite-reference inverse theorem, higher-order clock and propagation calculations, synthetic experiments, and limited analyses of published data. Appendix C gives the central equations and the calibration restrictions. Its scalar–matter coupling is denoted here by α_{ST}, to distinguish it from the normalization-orbit label in Chapter 10.

For a massless constant-coupling submodel, the source combines the Cassini summary for $\gamma - 1$ with one selected lunar-ranging summary. Under its uniform prior in $t = 1 - \gamma$, assumed zero cross-experiment covariance, and weak-body mapping, it reports $\left|\alpha_{ST}\right\rangle_{95} = 0.003959254$. This is a conditional summary-data posterior, not a bound on brane tension. The selected Ke et al. lunar result is itself conditioned on fixed ephemerides and uses conservative expanded uncertainty scales. Its alternative ephemeris reductions are not additional independent experiments to multiply into the same likelihood. [40] [42]

The same manuscript audits thirteen APOLLO normal-point records from two dates in August 2019. Parsing those records and converting their quoted timing uncertainties is useful provenance work. It does not fit a gravitational signal, derive a scalar range, or replace the orbital, clock, reflector, atmosphere, and station models needed to predict a ranging observation. Raw photon records and a complete observation normal matrix were not fitted in that calculation. [40]

13.5 A restricted DESI comparison

BAO measurements constrain combinations of cosmological distances and a sound-horizon scale. In a flat geometry the relevant model distances can be expressed as

$$\begin{aligned} \frac{D_H(z)}{r_d} &= \frac{A}{E(z)}, \\ \frac{D_M(z)}{r_d} &= A\int_0^z \frac{du}{E(u)}, \\ A = \frac{c}{H_0 r_d}&, E(z) = \frac{H(z)}{H_0}. \end{aligned} \tag{13.2}$$

Here r_d concerns the photon–baryon acoustic history. It is not a measured value of the dark-fluid function $dp/d\rho$ used in the brane inverse theorem. A free A also does not determine H_0 and r_d separately. [43]

The scalar–tensor source reports a calculation with a thirteen-component DESI DR2 compressed distance vector and its full covariance. For a shape vector $f(\vartheta)$ and covariance C, the fitted scale is obtained from

$$\begin{aligned} \chi^2(A,\vartheta) &= (d - Af)^T C^{-1} (d - Af), \\ \hat{A}(\vartheta) &= \frac{f^T C^{-1} d}{f^T C^{-1} f}. \end{aligned} \tag{13.3}$$

This profiling retains the correlations; summing squared residuals divided only by marginal variances would define a different statistic. The source describes manual transcription of the public inputs. Its local hashes do not establish equality with upstream bytes, and the supplied manuscript does not include the data and program needed for an independent rerun of that likelihood. The following numbers therefore retain their status as source-reported results. [40]

The native branch tested there is a flat, massless, zero-vacuum dust solution of the four-dimensional scalar–tensor model:

$$E(z) = (1+z)^{\nu}, \nu = \frac{3-\alpha_{ST}^2}{2(1-\alpha_{ST}^2)}, 0 \le \alpha_{ST}^2 < 1. \tag{13.4}$$

Its scalar-free limit has $\nu = 1.5$. The manuscript separately fits a free power-law control and a flat ΛCDM control. [40]

Table 13.1. Source-reported compressed-distance comparisons.

Distance-shape model	Reported shape result	Reported χ^2
Native scalar-free flat dust	$\nu = 1.5$ fixed	1420.183

Distance-shape model	Reported shape result	Reported χ^2
Native dust, $0 \le \alpha_{ST} \le 0.1$	Optimum at $\alpha_{ST} = 0$	1420.183
Independent power-law control	$\nu = 1.095896$	96.966
Separate flat ΛCDM control	$\Omega_m = 0.297462$	10.271

Each fit profiles the same free BAO scale. The large discrepancy of the native dust branch is a useful negative result for that branch under the compressed likelihood. Adding a vacuum component in the ΛCDM control changes the model. Its better fit cannot be transferred to the zero-vacuum branch, the full scalar–tensor action, or the five-dimensional inverse theorem. Nor does the extreme dust statistic justify an extrapolation to a spectacular far-tail significance without validating that use of the compressed likelihood. [40]

The predecessor's reuse of the scalar-free dust comparison is also the reuse of the same data and control, rather than independent confirmation. The final dynamical memorandum performs no DESI, Planck, supernova, or nuclear-reaction likelihood. Its hierarchy calculations address physical scales at supplied densities. [3] [4]

13.6 Reproducible calculations and an observational decision

The final memorandum reports thirty-three named checks, including the action-scaling identities, kinetic coefficients, acceleration equation, and numerical reconstruction of a prescribed nonconstant sound history. Its illustrative profile is $\chi(s) = 0.22 + 0.07 \sin(1.7\, s)$ on $[-1, 1]$, with $\rho(0) = 3$, $\rho(0) + p(0) = 2.6$, and $X(0) = 1$ in model units. The reported recovery error from independently differentiated interpolants is 5.92×10^{-8} in the interval interior. These are generated-model tests, not observations of a sound history. [4]

The book's direct evaluations of the scale formulas are distinct from those source-reported verification records. Reproducing the full source suite requires its programs, inputs, tolerances, environment, and output logs. A count of passed names demonstrates what the document claims; actual execution evidence requires the executable materials.

An observational extension must fit the recorded quantities with their calibration and trajectory nuisances. Signal injection should test attenuation; joint covariance should test agreement across channels. A scalar-free null has an unidentified range, so its detection statistic requires separate calibration. [40]

14 What Is Established and What Remains to Be Measured

14.1 The result is a hierarchy of identifiable quantities

The central result of this book is conditional but substantive. Within a declared five-dimensional Einstein-bulk model, an exact barotropic sound history converts cosmological reconstruction into a four-column inverse. When those columns are independent, the expansion fixes their coefficients. If the effective vacuum amplitude is free, the coefficients determine an equivalence class of matter normalizations and split tensions. The bulk curvature combination and geometric mass coefficient remain constant along that class. These are different levels of identification, and each has a precise set of hypotheses. [18; 19; 3]

The Riccati classification marks the exact boundary of the structural result. At an exceptional sound history, the dark-radiation column is already quadratic in the sound coordinate. Complete expansion and sound histories can then agree while the bulk metric changes. Away from the exception, carefully chosen finite epochs may identify coefficients, but small singular values can still make numerical reconstruction uninformative. Uniqueness, stability and measurement precision are separate accomplishments.

Table 14.1. Established results and the information they require.

Established statement	Information and model on which it depends
A sound coordinate linearizes the conserved density history	One conserved barotropic component and the supplied sound function
The Riccati family exhausts full-history column dependence	Exact differentiable sound on the whole stated interval
Free vacuum leaves a complete normalization family	Identified coefficients, physical interior and the declared reference convention
Bulk curvature and geometric mass survive that family	Rank four, positive quadratic coefficient and the AdS branch

Established statement	Information and model on which it depends
Finite uncertainties admit simultaneous outer confidence sets	A valid observation model, calibrated windows and a justified regularity class
A local scalar action can realize the history	Positive enthalpy and sound, over the visited field interval

This hierarchy explains why a geometric reconstruction can be informative without establishing a unique microscopic theory. It also makes failure interpretable. An inconsistent overdetermined design, incompatible anchors, or an empty properly calibrated confidence set rejects at least one premise of the combined model and measurement procedure; it need not identify which premise failed.

14.2 What the dynamical completion adds

The action analysis moves beyond curve reconstruction by identifying which freedoms have physical content. A conserved barotropic history with positive enthalpy and sound admits a local purely kinetic representation. The construction proves existence and supplies the homogeneous equation of motion. It does not select one kinetic law from all the laws that could reproduce the chosen history. Shift symmetry permits many functions of X and selects no golden-ratio frequency. [4]

Two normalization issues remain distinct. The change $\lambda_{bare} \mapsto \lambda_{bare} + \Delta$, $U_0 \mapsto U_0 - \Delta$ leaves the action identical because only $\tau_{vac} = \lambda_{bare} + U_0$ occurs. No observation of that action separates the two labels. The classical scaling orbit instead multiplies the action by a positive constant. It preserves the reference split coefficient but changes the vacuum-subtracted background coefficient in fixed stress units. A justified absolute measurement of the latter can select one orbit representative, while the exact split redundancy remains. [7; 3; 4]

The constant-sound example also demonstrates why a local action needs a dynamical audit. The pure cubic kinetic law has positive scalar kinetic coefficients at every strictly positive X, yet they vanish at its zero-gradient vacuum. It has no quadratic scalar propagator there. Adding a canonical term makes that vacuum locally regular but changes the sound speed:

$$F(X) = X + \frac{X^3}{M^8}, c_\phi^2 = \frac{1+3x^2}{1+15x^2}, x = \frac{X}{M^4}. \tag{14.1}$$

The sound approaches one fifth only at large x. If the polynomial is interpreted as an ordinary small-x expansion, that regime is not controlled without further justification. A finite-background effective theory could have a different cutoff, but positive sound speed alone neither supplies it nor proves coupled brane–bulk stability. The demonstrated repair resolves a specific local defect; it does not complete the ultraviolet theory. [4]

14.3 The direct late-time quadratic route is quantitatively restricted

Once the total constant vacuum stress and low-energy gravitational matching are declared, the direct quadratic-to-linear density ratio is

$$\epsilon_{brane}=\frac{\rho_e}{2\tau_{vac}}=\frac{H_\rho^2\ell^2}{4\left(1+\Lambda_{eff}\ell^2/3\right)},\ H_\rho^2=\frac{\rho_e}{3M_4^2}. \tag{14.2}$$

At the source's benchmark $H_0=70\,km\,s^{-1}\,M\,pc^{-1}$, $M_4=2.435\times10^{18}\,GeV$ and excitation density equal to the corresponding critical-density scale, a total tension scale $\tau_{vac}^{1/4}=1\,TeV$ gives $\epsilon_{brane}\simeq1.983\times10^{-59}$. This is a declared scale calculation. It provides no practical late-time expansion signal from this direct correction at those inputs. [4]

A one-percent ratio at the same density instead requires a tension scale of approximately $6.67\,meV$. With zero effective vacuum in the conversion, its AdS radius is approximately $0.86\,Gpc$. The resulting scale conflict concerns the standard localized-gravity interpretation. It does not show that all dark radiation, scalar clustering or higher-dimensional effects are too small.

The acceleration conclusion is comparably precise. Positive total tension, nonnegative excitation density and pressure, nonnegative dark radiation, and nonpositive effective vacuum give $\ddot{a}\le0$. The positive quadratic correction cannot accelerate that matter content. Adding a constant scalar vacuum may change the effective vacuum term, but does not turn the kinetic correction into a new source of negative pressure. Extensions with other matter stresses or different gravitational terms constitute different hypotheses. [4]

14.4 A measurement of the inverse requires its own forward model

The first observational task is to connect measured quantities to the information assumed by the inverse. Distances are not an exact table of H^2, and BAO

measurements do not supply a continuous dark-fluid $d\,p / d\,\rho$. Their acoustic scale involves the baryon–photon history. Extracting the dark-fluid sound information would require a perturbation model that specifies matter content, entropy and anisotropic stresses, its gravitational coupling, and the relevant bulk boundary conditions. The same model must propagate calibration and covariance into the sound windows and expansion coordinates. [15; 3; 4]

A concrete test can then be defined. With independently supplied exact sound and more than four suitable expansion epochs, let A be the four-column design and C the known Gaussian expansion covariance. In the unrestricted coefficient model,

$$T = \min_{\beta} (y - A\,\beta)^{T} C^{-1} (y - A\,\beta) \sim \chi^{2}_{N - \operatorname{rank} A}. \tag{14.3}$$

This checks the quadratic-plus-dark-radiation representation. Estimated sound, frequency selection, unknown covariance and physical-boundary fitting change the calibration; the procedures in the statistical chapters address particular versions of those problems. Rejecting the representation does not uniquely diagnose an extra dimension. Accepting it does not exclude a sufficiently flexible four-dimensional effective stress model. [27; 6; 3]

Absolute matter information gives a complementary test. Two calibrated density differences must select the same transport normalization, and calibrated enthalpy divided by Q must be constant. The vacuum convention and stress units must be explicit. Their uncertainties belong in the joint set, not in independent substitutions of central values. This would make the proposed reconstruction an observation-level inverse with stated alternatives; the supplied research does not yet contain that combined cosmological experiment.

14.5 Two physically concrete routes beyond the present model

One route is a local gravitational matching calculation for a fully normalized scalar sector. The action, its matter standards and a background solution define the problem. A localized source then determines the induced metric response on the brane, including the scalar and bulk contributions under declared boundary conditions. The resulting force and clock observables establish when the cosmological coefficient g_{bg} agrees with locally measured $8\pi G_N$. This calculation would test the operational calibration used in the hierarchy estimates and identify the parameter combinations actually fixed by local gravity. Its result cannot be

assumed for every rolling-scalar background merely because the background Friedmann equation has a familiar linear term. [4]

A second route is an early high-density model. For a radiation benchmark,

$$T(\epsilon)=\left(\frac{60\,\epsilon\,\tau_{vac}}{\pi^2 g_*}\right)^{1/4}. \tag{14.4}$$

At the TeV tension benchmark and fixed $g_*=106.75$, ratios 0.1 and 1 occur near 274.7 and $488.5\,GeV$. These temperatures locate a regime where the quadratic term is no longer negligible. They do not predict a spectrum. Bulk-graviton production and energy transfer can invalidate the vacuum-bulk and no-exchange assumptions, requiring an enlarged background and perturbation system. A nuclear-reaction or CMB calculation would then use that system, rather than importing the constant-tension inverse unchanged. [4]

Varying the tension also changes the conservation problem. In the limited scalar action where $U(\phi)$ and $\lambda_{bare}(\phi)$ enter only through their sum, renaming part of the potential adds no new degree of freedom. If matter excludes the variable tension, vacuum-bulk conservation reads

$$\dot{\rho}_m+3H(\rho_m+p_m)=-\dot{\lambda}_{bare}. \tag{14.5}$$

Imposing separate conservation of the same matter while varying that tension would contradict this equation unless additional transfer terms are included. A viable extension needs dynamics and observable couplings that distinguish its ingredients. [4]

14.6 Evidence, independent comparisons and scientific responsibility

The source calculations occupy several evidentiary levels: exact derivations, synthetic numerical checks, conditional analyses of observation summaries, and proposed applications. Counts of passed checks assess specified implementations; they are not counts of independent confirmations of a gravitational law. Reproducing a source table also differs from independently deriving its observation likelihood or comparing against new measurements. The book’s provenance and computational records preserve these distinctions.

The separate clock–lensing–dynamics study uses a four-dimensional massive scalar –tensor action. Its finite-reference inverse and source-calibration issues illuminate the same general problem of identifying parameters from observables. Its scalar range, charges and physical metric are not the brane tension, dark-radiation

coefficient or five-dimensional coupling. Its compressed-distance comparison tests a specified cosmological branch, not the bulk-identification theorem proved here. Keeping that study in a separate appendix makes the common inferential lesson useful without asserting an undeveloped unification. [40; 3]

A complete account can establish a mathematical result and document its physical limitations while leaving future measurements open. Responsible authorship requires the derivations, references, numerical status and claim boundaries to remain inspectable. AI assistance supplies no independent peer review, and a coherent synthesis does not certify historical priority. The meaningful next advance would be a calibrated observation or a dynamically justified restriction that excludes members of the present equivalence class for a stated physical reason. That advance has a defined target: identifying what nature fixes after the mathematical freedoms, measurement errors and action conventions have been accounted for.

Appendix A Units and Convention Crosswalk

A.1 Natural units and derivative conventions

Unless an SI conversion is stated, the book uses $c=\hbar=1$ and signature $(-,+,+,+)$. Write E for an energy unit; length and time then have units E^{-1}. The scale factor a has dimensions of length because the constant-curvature spatial coordinates have a fixed dimensionless normalization. A convention in which a is dimensionless requires a corresponding change in the curvature and bulk coefficients before numerical values can be compared.

The coordinate $s=\log(a/a_*)$ is dimensionless. A prime denotes its derivative, while a dot denotes a derivative with respect to cosmological proper time:

$$\frac{ds}{dt}=H,\ \dot{f}=Hf',\ \chi=\frac{dp}{dv}. \tag{A.1}$$

The logarithmic coordinate describes the chosen monotone branch of the cosmology. It is not a replacement for proper time at an expansion turning point. In Chapters 4–9, p is dimensionless pressure, equal to $\kappa_0 P/S$ for physical pressure P. Chapter 2 first denotes this same dimensionless variable by ϖ. Scalar pressures such as p_ϕ in the action analysis carry physical density units when written with their physical Lagrangian.

Table A.1. Dimensions in the braneworld convention. The scalar convention is $X=-h^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi/2$. A dimensionless logarithmic frequency is distinct from a physical fluctuation frequency, which has energy units. The symbol S in reconstruction is a curvature scale, whereas the full action is dimensionless in natural units.

Quantity	Symbol or definition	Natural units
Radius, scale factor and bulk curvature length	r,a,a_*,ℓ	E^{-1}
Hubble rate	H	E
Curvature scales	$S,\Lambda_5,\Lambda_4,\Lambda_{eff}$	E^2
Bulk metric mass coefficient	μ_B	E^{-2}
Reduced gravitational scales	M_5,M_4	E
Five-dimensional coupling	$G_5=\left(8\pi M_5^3\right)^{-1}$	E^{-3}

Quantity	Symbol or definition	Natural units
Four-dimensional reference or background coupling	$g_{ref}, g_{bg}, \kappa_0$	E^{-2}
Physical density, pressure and vacuum stresses	$\rho, P, \lambda, U_0, \tau$	E^4
Scalar field in the adopted convention	ϕ	E
Scalar kinetic variable and physical Lagrangian	$X, F(X)$	E^4
Dimensionless reconstruction quantities	$h, v, p, b, u, d, \eta, Q, J, L$	1
Logarithmic frequencies, widths and sound ratio	$\omega, \Delta, w, s, \chi$	1

A.2 Split tension and total vacuum stress

The bare or split tension is written λ or λ_{bare}. When the matter Lagrangian contains a constant contribution $-U_0$, the total vacuum stress is $\tau = \tau_{vac} = \lambda + U_0$. The two associated linear coefficients and constant expansion terms are

$$g_{ref} = g_{split} = \frac{\lambda}{6M_5^6}, g_{bg} = \frac{\tau}{6M_5^6}, \\ \frac{\Lambda_4}{3} = \frac{\lambda^2}{36M_5^6} - \ell^{-2}, \frac{\Lambda_{eff}}{3} = \frac{\tau^2}{36M_5^6} - \ell^{-2}. \tag{A.2}$$

The density accompanying g_{ref} may include U_0. The excitation density accompanying g_{bg} has that constant subtracted. Replacing Λ_4 by Λ_{eff} while retaining the former density and coupling is therefore not a consistent change of notation. All parts of the split must change together. [4]

The reference quantity $G_{4,ref}$ is defined by $8\pi G_{4,ref} = g_{ref}$. It is not automatically a measured Newton constant. Similarly, $M_4^{-2} = g_{bg}$ defines the vacuum-subtracted background scale; identifying it with local gravity requires the matching conditions discussed in Chapter 10.

A.3 Dimensionless reconstruction and physical conversion

With fixed S, a_* and $\kappa_0 = g_{ref}/3$, the normalization is

$$\begin{gathered} h = H^2/S,\ v = \kappa_0 \rho/S, \\ p = \kappa_0 P/S,\ b = \Lambda_4/(3S), \\ u = k/(a_*^2 S),\ d = \mu_B/(a_*^4 S), \\ \eta = \frac{S}{2\kappa_0 \lambda},\ L = \frac{1}{4\eta} - b. \end{gathered} \tag{A.3}$$

The symbol τ is reserved for vacuum stress; η^{-1} always means the inverse of the dimensionless inverse tension. Physical conversion gives

$$\begin{gathered} \lambda = \frac{S}{2\kappa_0 \eta},\ \ell = (S L)^{-1/2}, \\ \mu_B = a_*^4 S d,\ \rho = (S/\kappa_0) v. \end{gathered} \tag{A.4}$$

These expressions require positive finite η and positive L for the stated AdS branch. The limit $\eta = 0$ is not a finite positive-tension representative. The conversion of an identified coefficient into energy density also remains conditional on the supplied reference coupling and scales. Changing units is different from independently measuring those scales. [3]

For a local scalar completion written as $P(X) = p$, the physical pressure Lagrangian is $(S/\kappa_0) P(X)$. Naming a coefficient in this representation does not fix an absolute matter normalization. If a convention $F(0) = 0$ is used to isolate U_0, the action must first possess a well-defined extension to $X = 0$.

A.4 Coordinate normalization and SI restoration

In a planar bulk, a spatial-coordinate rescaling can be accompanied by

$$\begin{gathered} \tilde{r} = \zeta r,\ \tilde{a}_* = \zeta a_*, \\ \tilde{T} = T/\zeta,\ \tilde{x} = x/\zeta, \\ \tilde{\mu}_B = \zeta^4 \mu_B,\ \tilde{\ell} = \ell,\ \zeta > 0. \end{gathered} \tag{A.5}$$

Here x denotes the normalized planar coordinates. This relabeling preserves the geometry. The quantities d and r_h/a_* remain invariant at fixed S, while the numerical coefficient μ_B depends on the declared normalization. A physical bulk mass additionally involves G_5 and the spatial-volume convention. [3]

For SI formulas using energy density ε and tension in joules per cubic metre, restore

$$h=\frac{H^2}{c^2 S},\kappa_{0,SI}=\frac{8\pi G_{4,ref}}{3c^4},\nu=\frac{\kappa_{0,SI}\varepsilon}{S}. \tag{A.6}$$

The remaining dimensionless definitions retain their form with this SI coefficient. Using mass density instead requires its factor of c^2. In an effect estimate containing $H_\rho \ell$, an SI Hubble rate and a length must occur as $H_\rho \ell / c$. Sound means remain averages of the dimensionless χ in s; physical sound speed satisfies $c_s^2=c^2\chi$ under the stated barotropic interpretation. These conversions prevent a unit convention from being mistaken for a physical constraint.

Appendix B Reconstruction Algorithms

B.1 Choose the data problem

Record the interval, reference scales, supplied curvature and vacuum terms, matter class and calibration assumptions before solving an inverse. Distinguish an exact continuous history, finite exact values, deterministic error bounds and a statistical measurement region. Each supplies different information. A finite grid does not become a continuous band merely through interpolation. The procedures below specify mathematical targets; numerical use must record integration errors, rank conventions, physical-domain checks and solver statuses.

Use $s=\log\left(a/a_{*}\right)$ on an interval containing zero, with fixed S, a_{*}, κ_{0} and supplied u. Include positive density, positive enthalpy and the chosen expanding branch among the requirements. Treat $\eta=0$ as a limiting effective case, separately from finite positive tension.

B.2 Exact sound with supplied vacuum

Construct the transport from the supplied sound history:

$$Q(s)=\exp\left[-3s-3\int_{0}^{s}\chi(t)\,dt\right], J(s)=-\int_{0}^{s}Q(t)\,dt. \tag{B.1}$$

At at least four expansion epochs, assemble

$$A_{i}=\left(1, J_{i}, J_{i}^{2}, e^{-4s_{i}}\right), f_{i}=h_{i}-b+u\,e^{-2s_{i}}. \tag{B.2}$$

Check column rank and solve $A\theta=f$. With more than four observations, require consistency of the entire vector. If rank fails, retain the coefficient solution set; a pseudoinverse gives one representative rather than a unique physical solution. Small singular values quantify sensitivity in the declared scaling.

Use a factorization, such as QR or SVD, instead of explicitly inverting the matrix. Record the column scaling and threshold used to assess rank. A small singular value is numerical evidence about conditioning, not an analytic proof of exact dependence. With noisy values, a least-squares solution is a point estimate and belongs to the uncertainty route below rather than to the exact-consistency calculation.

For $\theta=(\alpha_0,\alpha_1,\alpha_2,d)$, use

$$\Delta_\alpha=\alpha_1^2-4\,\alpha_0\alpha_2,\,C=\sqrt{\Delta_\alpha},\quad \eta=\alpha_2/\Delta_\alpha,\,V=\frac{2\,C\,\alpha_0}{\alpha_1+C}. \tag{B.3}$$

Require positive $\alpha_0,\alpha_1,\alpha_2,\Delta_\alpha$, nonnegative d, $1/(4\,\eta)>b$, and positive $v=V+C\,J$ and h throughout the interval. Convert units only after these checks. Four consistent samples do not establish agreement with an independently supplied full curve. [18; 19]

B.3 Exact sound with free vacuum

Use the same matrix but solve against $y_i=h_i+u\,e^{-2s_i}$ for $(\beta_0,\beta_1,\beta_2,d)$. For an identified vector with $\beta_2>0$, calculate

$$\Sigma=\beta_1^2-4\,\beta_0\beta_2,\,L=\frac{\Sigma}{4\,\beta_2},\quad T(s)=\beta_1+2\,\beta_2\,J(s),\,C_{max}=T(s_+). \tag{B.4}$$

Require $L>0$, $C_{max}>0$, $d\ge 0$ and positive common expansion. Return the full family

$$0<C<C_{max},\,\eta_C=\frac{\beta_2}{C^2},\quad V_C=\frac{C(\beta_1-C)}{2\,\beta_2},\,b_C=\frac{C^2-\Sigma}{4\,\beta_2}. \tag{B.5}$$

Report $\ell=(S\,L)^{-1/2}$ and $\mu_B=a_*^4\,S\,d$ with their normalization. Additional priors restrict the family explicitly. One density anchor requires solving a quadratic and retaining every admissible root. Two distinct density anchors or an absolute enthalpy anchor determine a candidate C, which must still fit all observations. Never apply the fixed-vacuum discriminant inverse after freeing the intercept. [3]

B.4 Finite values with a common matter history

For this route, supply b,u,d and consider each candidate η. Transform positive residual intervals using

$$R_i=h_i-b+u\,e^{-2s_i}-d\,e^{-4s_i},\,f_\eta(R)=\frac{2\,R}{1+\sqrt{1+4\,\eta\,R}}. \tag{B.6}$$

Use the union of density epochs and sound-window endpoints as the node set. Introduce one density v_i and one rate $q_i>0$ at every node, shared by both adjacent cells. For a sound margin $0\le m<1/2$, set $A_m=3(1+m)$ and $B_m=3(2-m)$. On each cell of length Δ, impose

$$\begin{aligned} e^{-B_m\Delta}q_i &\le q_{i+1}\le e^{-A_m\Delta}q_i,\\ q_-(t)&=max\left(q_i e^{-B_m t}, q_{i+1}e^{A_m(\Delta-t)}\right),\\ q_+(t)&=min\left(q_i e^{-A_m t}, q_{i+1}e^{B_m(\Delta-t)}\right),\\ \int_0^\Delta q_- dt &\le v_i - v_{i+1}\le \int_0^\Delta q_+ dt. \end{aligned} \tag{B.7}$$

The exponential branches integrate exactly. Chapter 4 gives their cone representations. Add sound means as endpoint-rate ratios using (4.19), and impose all proxy-gain inequalities on these same densities. With logarithmic proxy bounds $y^{\pm}_{p,i}$, these are

$$v_j\le \exp\{\delta|s_i-s_j) - y^-_{p,i}+y^+_{p,j}\}v_i \text{ for every } i,j. \tag{B.8}$$

The sufficient background condition $b-ue^{-2s}+de^{-4s}\ge 0$ guarantees positive expansion between nodes. Without it, positivity needs an additional justified check. A positive first-to-last density separation excludes the constant zero-enthalpy branch.

Return feasibility with a common witness, infeasibility with appropriate evidence, or an unresolved status. Do not interpolate untested parameter gaps or convert solver failure into exclusion. Interval endpoints are justified only where an interval theorem applies. A smooth completion preserves finite constraints with a smaller positive margin; it does not automatically preserve an added derivative budget. [11; 16; 15]

B.5 Propagate a common uncertainty event

Obtain a simultaneous measurement region from the actual observation model. Independent complete Gaussian repeats permit the coordinate Student construction in Chapter 9. Distinct epochs are not substitutes for those repeats. Declare a regularity bound covering the entire sound class and the full windows before deriving a uniform error ε_χ.

Integrate the nominal sound curve with justified errors ζ_i for numerical values $\widetilde{J}_i$. Form

$$\begin{aligned} R_i &= \left(\left|\widetilde{J}_i\right| + \zeta_i\right)\left(e^{3\varepsilon_\chi |s_i|} - 1\right) + \zeta_i, \\ \widehat{A}_i &= \left(1, \widetilde{J}_i, \widetilde{J}_i^2, E_i\right), \\ B_i &= \left(0, R_i, 2\left|\widetilde{J}_i\right| R_i + R_i^2, 0\right). \end{aligned} \tag{B.9}$$

For fixed vacuum, retain every nonnegative coefficient vector satisfying

$$\left|\widehat{A}\,\theta - f\right| \le e + B\,\theta. \tag{B.10}$$

For free vacuum, replace f by y and allow the intercept to have either sign. Its uncertainty column is zero, so the inequalities remain linear when the other coefficients are nonnegative. The result is an outer set. Enforce physical constraints through all compatible sound histories, not only the nominal curve, and map every surviving free-vacuum C. Report unbounded outputs as unbounded rather than selecting convenient finite endpoints. [20; 6]

For at least four expansion epochs and a positive diagonal 4×4 scaling W, a useful rank enclosure is

$$\gamma = \sigma_{min}\left(\widehat{A}\,W\right) - \left\| B\,W \right\|_F. \tag{B.11}$$

A positive value certifies full column rank for every matrix admitted by the sound bound. A nonpositive value is inconclusive. This check addresses matrix uncertainty; it does not make all permitted histories yield one coefficient vector or remove the free-vacuum family. Account for numerical error in both terms if the reported rank enclosure is intended as a rigorous computational certificate.

B.6 Report the scope of the result

A useful reconstruction record includes the supplied nuisance quantities, measurement event, regularity budget, retained coefficient set, physical conversion and unresolved cases. Distinguish exact identities from floating-point comparisons and distinguish common-history witnesses from outer enclosures. If continuous weak integrals are used, supply continuous expansion and sound bands; four expansion values are insufficient. If a mode was selected statistically, identify the calibrated test and its actual target separately from the physical confidence set. This record makes the result reproducible and exposes which additional measurement would narrow the remaining ambiguity.

Appendix C Scalar–tensor comparison and finite clock references

C.1 A separate four-dimensional model

The scalar–tensor source provides a useful comparison because it develops an explicit inference problem with finite clocks, lensing, and massive tracers. Its action is a four-dimensional canonical scalar coupled universally through a conformal matter metric. With $x^0 = c\,t$, it can be written

$$S_{ST} = \frac{c^3}{16\pi G_0}\int d^4x\sqrt{-g}\left[R - 2(\partial\varphi)^2 - 2\mu_\varphi^2\varphi^2\right] + S_m\left[\tilde{g}, \psi\right), \tilde{g}_{\mu\nu} = e^{2\alpha_{ST}\varphi} g_{\mu\nu}. \tag{C.1}$$

Here φ and α_{ST} are dimensionless, while μ_φ is an inverse length. The source's linear nonmetricity representation is equivalent to this curvature action under its declared matter and boundary conventions. It is an established scalar–tensor class, not an independent observational identification of nonmetricity. [40]

Neither G_0 nor the scalar range μ_φ^{-1} is a renamed five-dimensional coupling or brane tension. A common action embedding these sectors has not been supplied. The comparison concerns the logic of inference and calibration.

C.2 Weak-source potentials and a common mass standard

For a weak, isolated, spherical source, the first-order physical potentials are

$$U(r) = \frac{K}{r}\left[1 + B\,e^{-\mu_\varphi r}\right), \quad W(r) = \frac{K}{r}\left[1 - B\,e^{-\mu_\varphi r}\right), \quad K = G_0 M, B = \alpha_{ST}^2 F\left(\mu_\varphi\right). \tag{C.2}$$

The form factor F depends on the source profile. The sum $U + W = 2K/r$ removes the direct conformal scalar from first-order bending at fixed tensor mass. Lensing supplies $K_L = c^2 b_L \theta/4$ at this order, with b_L the light-ray impact parameter, while material acceleration contains the factor $1 + B\left(1 + \mu_\varphi r\right)e^{-\mu_\varphi r}$. A mass already inferred from material motion is not automatically the same calibration. [40]

In the formulas below, a_r is the positive inward acceleration magnitude in this static weak-field model. Kinematic, multipole, and environmental contributions

must be handled before interpreting a measured trajectory through that radial quantity.

The approximation order matters. In the source's Solar-scale example, omission of the second-order Schwarzschild bending term creates a 6.25129 ppm mass bias, larger than its assigned 2.94304 ppm scalar dynamical residual. This is a model error-budget example, not a detected signal. Its purpose is to require a sufficiently accurate baseline before applying the inverse.

C.3 Finite-reference identification

Let stationary clocks occupy exterior radii $a<b$. Write $\eta=b/a>1$, $x=\mu_\varphi a$, and let $R_{ab}=d\tilde{\tau}_a/d\tilde{\tau}_b$ be their corrected rate ratio. Measured photon-frequency transfer must first be related to that orientation. Define

$$Z_{ab}=\frac{a\,c^2}{K_L}\left(1-R_{ab}\right)-\left(1-\frac{a}{b}\right),$$
$$D_a=\frac{a^2 a_r(a)}{K_L}-1. \tag{C.3}$$

At the retained order, $Z_{ab}=B\left(e^{-x}-\eta^{-1}e^{-\eta x}\right)$ and $D_a=B(1+x)e^{-x}$. The common amplitude cancels:

$$\frac{Z_{ab}}{D_a}=h_\eta(x)=\frac{1-\eta^{-1}e^{-(\eta-1)x}}{1+x},$$
$$B=\frac{D_a e^{x}}{1+x},\ \mu_\varphi=\frac{x}{a}. \tag{C.4}$$

Finite-reference proposition. Given a correctly shared K_L, a resolved $D_a>0$, and the isolated single-range model, a noiseless ratio in $0<Z_{ab}/D_a\le 1-1/\eta$ identifies a unique finite $x\ge 0$. [40]

Proof. The function starts at $1-1/\eta$ and tends to zero. Its derivative has numerator $e^{-(\eta-1)x}\left[1+(\eta-1)x/\eta\right)-1$, which is negative for $x>0$. It is therefore strictly decreasing away from the endpoint. The amplitude then follows from Equation C.4.

At a scalar-free null both residuals vanish and the range is unidentified. Near the long-range endpoint, the expansion is $h_\eta(x)=1-1/\eta-(\eta-1)^2x^2/(2\eta)+O\left(x^3\right)$: the linear range response cancels. These are structural limits, not defects repaired by collecting repeated measurements at effectively one radius. Noisy ratios near either boundary require a forward fit to the original observables and their covariance.

C.4 A derivative identity is not a unique signature

For residuals defined from a differentiable potential and its force,

$$C(r)=\frac{r\,U(r)}{K_L}-1,\ D(r)=\frac{r^2 a_r(r)}{K_L}-1,\qquad D=C-r\,C'. \tag{C.5}$$

The last relation follows from differentiation for any such potential. A single exponential requires additional conditions, including a constant logarithmic slope and agreement across radii. A positive sum of several exponentials satisfies the derivative identity but generally has a changing slope. [40]

A common calibration error $\widehat{K}_L=(1+\delta)K_L$ gives

$$\widehat{C}=\frac{C-\delta}{1+\delta},\ \widehat{D}=\frac{D-\delta}{1+\delta},\qquad \hat{\mu}_\varphi=\mu_\varphi\frac{C}{C-\delta}. \tag{C.6}$$

The generic derivative identity can survive that bias. Correct identification therefore requires calibration information or nuisance fitting in addition to the closure. This is the methodological connection with the brane normalization problem; it does not equate their actions.

Source structure remains relevant after the exponential range and amplitude have been inferred. For a uniform sphere of radius R,

$$F(z)=\frac{3(z\cosh z-\sinh z)}{z^3},\ z=\mu_\varphi R,\ F(0)=1. \tag{C.7}$$

A general positive spherical density supported inside the same radius obeys $1\le F\le \sinh z/z$. The measured B therefore restricts the microscopic coupling to $B\,z/\sinh z\le\alpha_{ST}^2\le B$ until the density profile is specified. It does not determine the sign of that coupling, which changes together with the sign convention for the scalar field. Finite test bodies can introduce an additional source factor. The inverse proposition identifies an exterior response under its assumptions, while microscopic interpretation needs these further source inputs. [40]

This is also why the clock and dynamical channels should be fitted together at the observable level. A shared reference clock contributes correlated errors to several links, and the same lens calibration can affect all inferred residuals. Treating every derived residual as independent would assign information that the measurement design has not supplied. A forward covariance model can retain that shared

information and fit the nuisance mass without taking unstable ratios near the scalar-free boundary. The source's synthetic studies illustrate such a design; they do not determine the covariance of a future experiment. [40]

C.5 Higher-order clocks and a stellar precision limit

The source completes a specified second-order static exterior clock problem using a scalar amplitude normalized at a finite matching radius. It combines the scalar correction with the physical metric, finite frequency transfer, and reciprocal ranging terms. The relevant special-function particular solution has an explicit antecedent in Schärer et al.; the finite-boundary convention and integration into the source's calculation should retain that attribution. [40] [44]

An exterior amplitude must be fitted as boundary information or obtained by an interior source calculation. Its leading weak-source expression cannot be imposed as an exact strong-body relation. Likewise, isotropic and areal radial coordinates must be converted before comparing second-order clock coefficients with a Schwarzschild expression used elsewhere in the book.

The stellar examples expose another precision boundary. The scalar–tensor manuscript reports thirty fully coupled static configurations with two published candidate-EOS approximations. At 1.4 nominal solar masses, its GR radii are 11.70606 km for SLy and 11.31968 km for AP4. Read et al.'s rounded tabulated radii and fit errors imply approximately 11.7113544 and 11.325148 km, differences of 5.2944 and 5.468 m. The source acknowledges that this external comparison has not been reconciled across all conventions. [40] [45]

By comparison, two fixed-central-density massless examples with $\alpha_{ST} = 0.002$ change their physical radii by only 1.12 and 0.24 mm. These use different matching conditions from the external benchmark; the comparison illustrates the precision scale, not a statistical error distribution. Tiny internal refinement changes cannot establish observable stellar constraints when the external and EOS uncertainties remain unresolved. Static convergence also does not supply radial stability, tidal response, or a population likelihood.

The source's tables and plots remain reported computational evidence; its executable archive was unavailable. The finite-reference proposition is a mathematical result under its assumptions. Further observational claims require the corresponding measurement and numerical support.

Appendix D Reproducible Calculations

D.1 What the companion calculation contains

The companion program `reproduce_book.py` generates the eight scientific figures in this edition, their tabulated data and a machine-readable verification record. Its inputs are the declared analytical examples and benchmark constants printed in the book. It uses no observational catalogue. The program can be run in a directory with Python, NumPy, SciPy, SymPy and Matplotlib installed:

```
python reproduce_book.py --output-dir results
```

The output contains a `figures` directory with PNG, PDF and SVG versions; a `data` directory with CSV tables; and `verification.json` with inputs, software versions and individual comparison results. The accompanying README states the run command and evidentiary scope. Vector figure exports support resizing without losing mathematical labels.

The executable checks cover symbolic identities and independent numerical comparisons. The 25 specified checks passed in the run used for this edition. They verify the named expressions or examples; they are not independent empirical confirmations of a cosmological model. The program stops on a failed check. Passing a floating-point tolerance certifies only the declared computation at that tolerance.

D.2 Algebra and evolution checks

The symbolic checks include the constant vacuum split, action-orbit scaling of the two gravitational coefficients, the acceleration equation, the bulk-curvature effect-size relation, the cubic and canonical-plus-cubic energy densities, the cubic quadratic and cubic fluctuation actions, current conservation, the fixed- and free-vacuum discriminants, the stable density inverse, the bulk vertex invariant, the Riccati exceptional family and the paired-window identity.

The canonical-plus-cubic example is computed by two different routes. One integrates the differential equation for $\log x(s)$ from $x(0)=1$. The other solves the conserved-current equation independently at the sampled epochs. Over $-3 \le s \le 3$, their largest absolute difference in $\log x$ is approximately 1.48×10^{-11} in the recorded run, and the largest relative current drift is approximately 1.91×10^{-11}. These diagnostics concern that background evolution only. They do not test bulk perturbations or an ultraviolet cutoff.

The equivalence-class check uses the exact Chapter 6 benchmark, including $k=u=0$, $(\beta_0,\beta_1,\beta_2,d)=(9.5632,3.216,0.27,10^{-4})$, $\chi=0.2$ and $-0.5\le s\le 0.5$. Three distinct normalizations reproduce the same expansion on the numerical comparison grid. The two values $C=0.216$ and $C=3$ also share $v(0)=1.2$. Numerical agreement illustrates the algebraic identity; the continuum result follows from the theorem, not from the finite grid.

The covariance figure compares an exact analytical plug-in rejection probability against a separately evaluated F survival function. It is not a new Monte Carlo ensemble. The rank-two calculation assumes independent Gaussian validation vectors and the ordinary unbiased covariance estimator. Its matched Hotelling curve concerns the same experiment.

D.3 Physical-scale inputs and results

The scale calculation fixes $H_0=70\,km\,s^{-1}\,Mpc^{-1}$ and $M_4=2.435\times10^{18}\,GeV$. It uses $1\,Mpc=3.0856775814913673\times10^{22}\,m$, $\hbar=6.582119569\times10^{-25}\,GeV\,s$ and $\hbar c=1.973269804\times10^{-16}\,GeV\,m$. These are explicit conversion inputs. The resulting reference density is $3.96594\times10^{-47}\,GeV^4$.

Table D.1. Declared physical-scale calculations.

Declared calculation	Result rounded for reading
Present reference-density ratio at $\tau_{vac}^{1/4}=1\,TeV$	$\epsilon_{brane}=1.98297\times10^{-59}$
Corresponding upper bound on $(H-H_{lin})/H_{lin}$ under Chapter 12's assumptions	9.91484×10^{-60}
Radiation ratio at $T=1\,MeV$, $g_*=10.75$, TeV tension	1.76830×10^{-24}
Tension scale giving radiation ratio 0.1 at that MeV benchmark	$2.05064\,MeV$
Temperature for ratio 0.1, TeV tension and fixed $g_*=106.75$	$274.708\,GeV$
Temperature for ratio 1 under the same inputs	$488.507\,GeV$

Declared calculation	Result rounded for reading
AdS radius at TeV tension and $\Lambda_{eff}=0$	$1.17696\,mm$
Tension scale corresponding to $\ell=0.1\,mm$	$3.43068\,TeV$
Tension scale for a present reference-density ratio of 0.01	$6.67312\,meV$
AdS radius for the preceding one-percent example	$0.856550\,Gpc$

The MeV threshold is a scale diagnostic at a stipulated ratio, not a nucleosynthesis likelihood bound. The millimetre conversion uses a specified zero-effective-vacuum relation, not a direct fit of a gravity experiment. The percent correction uses the reference excitation density and does not represent a measured cosmological deviation. Chapter 12 gives the assumptions that connect these calculations to the model.

D.4 Figure provenance and interpretation

Table D.2. Provenance of the scientific figures.

Figure	Calculation and interpretation
1.1	Constant-curvature triangle relation; a geometrical illustration
3.1	QR projection of an exponential design on a fixed 41-epoch layout; conditional information with known noise
4.1	Exact reachable-rate envelopes and the incompatible adjacent-triple example
6.1	The free-vacuum equivalence class from the explicit Chapter 6 coefficients
7.1	Uniform-window transfer functions over a declared finite frequency band
8.1	Exact Gaussian covariance-estimation calibration diagnostic
11.1	Canonical-plus-cubic sound and two-route homogeneous evolution
12.1	Direct quadratic-to-linear ratios from the stated physical benchmarks

All scientific curves in these figures are newly generated for this edition from the accompanying program. They are labeled examples or scale calculations. The cover illustration is conceptual artwork and does not encode a numerical spacetime solution.

D.5 Earlier numerical results

Some earlier manuscripts report synthetic experiments, nonlinear recovery errors, likelihood comparisons and larger internal verification totals. Those numbers remain attributed to the relevant source in the chapters. The original executable archives and observational analysis pipelines for those studies were not part of the material available for this edition. Their full historical runs were therefore not reproduced here. In particular, this companion does not recreate the source posterior calculations, the full compressed-distance analysis, the reported 33-check suite, or every simulation underlying Series II.

An independent reproduction of those results would require the original inputs, units, model branch, solver settings, random seeds where applicable and full covariance or likelihood specification. The companion supplied here is complete for the eight book figures and its own 25 specified checks. That narrower claim is directly inspectable.

Appendix E Source Register and Result Map

E.1 The manuscript sequence

This edition draws on 27 supplied research manuscripts: twelve articles in the first collected volume, twelve separate Series II papers, a separate four-dimensional scalar–tensor study, the predecessor bulk-geometry paper and the final dynamical-completion paper. The source collection thus contains both a sustained braneworld inverse program and an independent comparison model. A source label identifies a manuscript; it does not imply journal publication or peer review. Published primary references are listed separately by their bibliographic records in the References.

The book reorganizes the material by logical dependency. It does not present the sequence of source papers as a sequence of independent empirical confirmations. Repeated formulas and synthetic examples are consolidated, conventions are reconciled, and distinct assumptions are retained where the source problems differ.

Table E.1. Source manuscripts and their destinations.

Source	Main contribution used in the book	Main destination
Series I, I	Finite geometrical information and the operational inverse	Chapters 1–3
Series I, II	Finite-epoch expansion design and conditioning	Chapter 3
Series I, III	Matter calibration and normalization freedom	Chapters 3–4
Series I, IV	Bounded calibration drift and compatible gains	Chapter 4
Series I, V	Complete-curve matter restrictions	Chapter 4
Series I, VI	Shared finite histories and exact counterexamples	Chapter 4
Series I, VII	Smooth completion and conic representations	Chapter 4
Series I, VIII	Point sound anchors and regularity limits	Chapter 4
Series I, IX	Integrated sound windows and common reconstruction	Chapter 4
Series I, X	Sound coordinates and column dependence	Chapters 5–6
Series I, XI	Exceptional Riccati sound histories	Chapters 5–6
Series I, XII	Synthesis of the first inverse sequence	Chapters 1–6 and 14

Source	Main contribution used in the book	Main destination
Series II, I	A stipulated golden-scale sound family	Chapter 7
Series II, II	Window transforms, offset and gain calibration	Chapter 7
Series II, III	Algebraic recovery of two frequencies	Chapter 7
Series II, IV	Blind spots and a third window	Chapter 7
Series II, V	Joint sound–expansion local rank	Chapter 7
Series II, VI	Weak information and observational design	Chapter 7
Series II, VII	Shared-parameter goodness-of-fit calibration	Chapter 8
Series II, VIII	Selected-space validation and leakage	Chapter 8
Series II, IX	Conservative finite-sample regions	Chapter 8
Series II, X	Unknown covariance and validation replicates	Chapter 8
Series II, XI	Uniform sound uncertainty and outer inversion	Chapter 9
Series II, XII	Statistical synthesis and scope corrections	Chapters 8–9 and 14
Predecessor bulk paper	Free-vacuum equivalence classes and bulk invariants	Chapter 6
Final dynamical paper	Covariant normalization, kinetic audit and scale limits	Chapters 10–14
Separate scalar–tensor study	Clock, lensing and dynamics comparison in another action	Appendix C

The complete bibliographic titles appear in the References. [1; 2; 12; 10; 13; 11; 16; 17; 15; 18; 19; 5; 21; 23; 22; 24; 25; 26; 27; 28; 29; 30; 20; 6; 3; 4; 40]

E.2 Relation to the final dynamical paper

The final paper, *Dynamical Completion and Observational Limits of Identifiable Bulk Geometry*, supplies the principal action-level development. Its 26 numbered equations are incorporated through the covariant and normalization discussion of Chapter 10, the kinetic construction and fluctuation audit of Chapter 11, and the hierarchy, acceleration and varying-tension analysis of Chapter 12. The observation interfaces are developed in Chapter 13 and the combined conclusions in Chapter 14. The book also restores the earlier reconstruction results needed to understand what that final paper is completing.

The most consequential convention is the separation between the split reference coefficient and the vacuum-subtracted background coefficient. The background

coefficient depends on the total constant vacuum stress: moving a constant contribution between the split brane term and the matter vacuum leaves that total and the background coefficient unchanged. The distinct classical scaling of Chapter 10 changes the background coefficient while preserving the split reference coefficient. The direct quadratic-to-linear correction therefore uses the total constant vacuum stress after the vacuum subtraction has been specified. A corresponding local Newton coupling requires a localized-source matching calculation under the chosen matter standards.

This edition adds an integrated notation guide, explicit links between chapters, worked interpretation of the normalization family, new plots and their tabulated data, separate front and back matter, and a computational record. These additions are exposition and reproducibility work. They do not by themselves establish historical priority for a new physical law.

E.3 Verification and documentary boundaries

The scientific review checked the final source against the relevant earlier derivations, examined sign and normalization conventions, and compared the book's central formulas across chapters. It explicitly separates full-history results from finite sampling, structural rank from numerical conditioning, and local scalar stability from stability of the coupled brane–bulk system. It also retains the source's reported statistical shortcomings when later diagnostic experiments address a different question.

The References use supplied-document status for the research manuscripts unless an independently identifiable publication record is available. The book author is Mohammed F. A. Alghenaim. Bibliographic authorship of earlier supplied sources is retained only when their documents explicitly establish it. No academic affiliation, degree, institutional endorsement or biographical history has been inferred for this edition.

Glossary

Action normalization. The overall scale multiplying an action. A constant positive scale leaves classical stationary equations unchanged but need not leave quantum predictions unchanged at fixed $\hbar$.

AdS radius. The length ℓ setting the negative bulk curvature in the stated Einstein-bulk model.

Anchor. An independent matter value or difference used to constrain a reconstruction family. Its units and calibration are part of the input.

Barotropic fluid. A fluid with pressure specified as a function of its density on the relevant branch. Additional entropy or anisotropic-stress degrees of freedom require a broader description.

Bulk. The higher-dimensional spacetime in which the brane is embedded.

Calibration drift. An allowed change of a measurement gain across epochs, restricted here by a stated Lipschitz bound on log gain.

Classical scaling orbit. A parameter transformation that multiplies the complete specified action by one positive constant.

Common history. A single matter evolution satisfying all finite values, windows, conservation and calibration restrictions together.

Conditional inverse. A reconstruction made while declared nuisance parameters or background terms are held fixed.

Confidence region. A random set whose repeated-sampling coverage is guaranteed under a stated observation experiment and model class.

Dark radiation. The a^{-4} term induced by the geometric bulk mass coefficient in the homogeneous background; the name does not identify it with a measured thermal species.

Effective vacuum. The constant term in the chosen background Friedmann representation after its convention has been stated.

Enthalpy density. The sum of density and pressure; it determines the density decline under conservation in an expanding universe.

Equivalence class. A collection of distinct parameter descriptions yielding the same declared information. Equivalence is always relative to what is observed and to the model.

Identifiability. Uniqueness of a target quantity from ideal information within a declared class. It is distinct from the precision attainable with finite noisy data.

Information projection. Removal of nuisance directions from a local signal derivative using the specified noise metric.

Kinetic completion. A scalar pressure Lagrangian constructed to realize a specified conserved barotropic history on its visited kinetic interval.

Leakage. A residual projection of an allowed primary signal into a selected space used to search for an additional component.

Local stability. Positivity of the relevant quadratic kinetic and gradient coefficients on a specified background; it does not establish all-sector stability.

Look-elsewhere effect. The change in a test's null behavior when a frequency or other parameter is selected by a search over the data.

Outer confidence set. A set guaranteed to include the target identified set on a common coverage event, possibly also including points excluded by a sharper joint model.

Physical mass. A mass obtained after fixing the gravitational normalization, distinct from a metric's geometric mass coefficient.

Point identification. Identification of one value, in contrast to identification of a nontrivial set or continuous family.

Riccati exception. The classified sound histories for which the dark-radiation function belongs to the quadratic span of the sound coordinate.

Selection. A data-dependent choice of a frequency, direction or model component. Its use in later inference requires an appropriate calibration.

Sound coordinate. The integrated transport function J that turns a conserved density into the affine form $v = V + C\,J$.

Sound window. A specified average of the sound history over an interval, including the coordinate and weighting used in the average.

Split brane tension. The constant assigned to the brane term separately from a constant matter vacuum in a chosen decomposition.

Structural rank. Independence of the design functions or columns in exact arithmetic, without a claim about their numerical conditioning.

Total vacuum stress. The sum of the split brane constant and constant matter vacuum; this sum enters the vacuum-subtracted background normalization.

Training and validation. Independent data blocks used respectively to select a target and to perform a calibrated check of it.

Uniform coverage. A probability guarantee applying throughout the declared parameter or function class; it depends on every stated bound being valid.

Vacuum subtraction. Rewriting the stress as a constant vacuum contribution plus excitations, with the resulting background coefficients transformed consistently.

Window blind spot. A frequency at which a measurement window has zero response. Several windows can reduce exact blind spots without ensuring stability at every frequency.

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The supplied research manuscripts are distinguished below from published primary literature. Reference numbers follow first appearance in the book.

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[18] Brane tension and bulk black hole identifiability: Exact degeneracies and reconstruction from cosmological sound data. Unpublished research manuscript, Series I, Paper X, 30 September 2026.

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What does a cosmological history actually determine?

A successful reconstruction can fix a geometry while leaving the matter normalization uncertain. This book follows that possibility from finite geometrical information to sound coordinates, exact equivalence classes and a covariant action.

Its central braneworld inverse identifies the assumptions under which bulk curvature and a geometric mass coefficient survive a freedom in vacuum and density normalization. It then examines the obstacles that real inference must address: common matter histories, finite sound windows, frequency selection, uncertain covariance, local scalar dynamics and physical scale matching.

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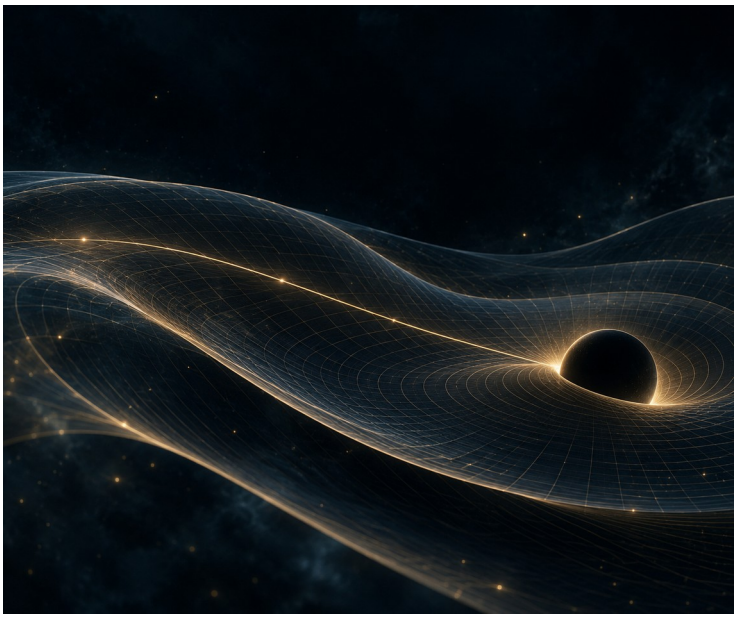

Mohammed F. A. Alghenaim